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RESEARCH REPORT No. BR-26

The Coexistence Problem for Hill's Equation

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Air Force Office of Scientific Research
Contract No. AF 49(638)-229
Project No. 47500

JULY, 1958

NEW YORK UNIVERSITY
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New York, 1958

Abstract

Hill's equation with a periodic coefficient of period π is investigated. The problem of the coexistence of two linearly independent solutions with period π or 2π is being considered, and all known cases in which such a coexistence occurs are shown to be special cases of a four-parametric equation which can be transformed into an equation of Hill's type. Also, the question of the existence of solutions admitting a terminating Fourier expansion is being investigated, and partial answers are being provided.

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I. Introduction

1. Statement of the problem

The general theory of differential equations with periodic coefficients dates back to Hermite (1877) and Picard (1879) for doubly-periodic coefficients, and Floquet (1883) for singly-period coefficients (see [26], p.375). Although a differential equation with periodic coefficients does not of necessity possess periodic solutions, it can be expected that for some values of the parameters of the equation such solutions will be obtained. The equation of second-order which appears to be most frequently treated is

$$(1) \qquad y'' + (h + \phi(x))y = 0$$

$$\text{with} \qquad \phi(x+\pi) = \phi(x) \; ; \qquad \int_0^\pi \phi(x)dx = 0 \; .$$

Equation (1) is the Liouville normal form of the classical Sturm-Liouville equation* and includes the well-known equations of Mathieu and Hill. The question of stability of the solutions, which was initiated by Liapounov and Poincaré, is of considerable interest. It turns out that there are infinite sequences of characteristic values of h

$$(2) \qquad h_0 < h_1 \leq h_2 < h_3 \leq h_4 < h_5 \leq h_6 < \dots$$

$$(3) \qquad \bar{h}_1 \leq \bar{h}_2 < \bar{h}_3 \leq \bar{h}_4 < \bar{h}_5 \leq \bar{h}_6 < \dots$$

such that the intervals

$$(4) \qquad (-\infty, h_0), (\bar{h}_1, \bar{h}_2), (h_1, h_2), (\bar{h}_3, \bar{h}_4), (h_3, h_4), \dots$$

---

* The transformation equations are given in [39] and [27].

are intervals of instability, and for values of h in these intervals there exist solutions of (1) which are unbounded as x approaches infinity[11], [32].

The values of h given in (2) are obtained from the standard solutions $y_1(x,h)$ and $1/2(x,h)$ which are defined by the initial conditions

$$y_1(0,h) = 1 \qquad y_1'(0,h) = 0$$

$$y_2(0,h) = 0 \qquad y_2'(0,h) = 0$$

by solving the equation

$$(5) \qquad y_1(\pi,h) + y_2'(\pi,h) - 2 = 0 \ ,$$

while those of (3) are obtained from

$$(6) \qquad y_1(\pi,h) + y_2'(\pi,h) + 2 = 0$$

and both sequences are connected by the inequalities

$$(7) \qquad -\infty < h_0 < \bar{h}_1 \leq \bar{h}_2 < h_1 \leq h_2 < \bar{h}_3 \leq \bar{h}_4 < h_3 \leq h_4 < \cdots$$

Thus the solutions with the fundamental periods π and 2π play a special role in the stability question since the h_n of (2) are the characteristic values for the solutions of period π, and the $\bar{h}_n$ of (3) are the characteristic values for the solutions of period 2π. If the intervals of (4) are numbered 0, 1, 2, 3, 4, etc., then the boundaries of the even intervals of instability correspond to solutions of period π while the boundaries of the odd intervals correspond to solutions of period 2π.

The present investigation arose from the following two questions:

(i) Are there other equations besides the Lamé equation for which all but a finite number of intervals of instability vanish.

(ii) For equations in which $\phi(x)$ is a finite polynomial in cos 2x (e.g., the Mathieu equation), will any of the intervals of instability vanish.

These questions are answered by considering the simultaneous occurrence of two linearly independent solutions of period π or 2π. Such solutions occur whenever two characteristic values of h coalesce and are always even and odd solutions of the same period. Details of the existing theory and previous work may be found in the recent review of Hill's equation by Magnus and Shenitzer[32].

2. Review of previous work

Perhaps the best known result in this field is the impossibility of the coexistence of two Mathieu functions; that is, two linearly independent solutions of the Mathieu equation with period π or 2π will not coexist. This was first proved by Ince[18] and also subsequently by Hille[15] and Markovic[33]. This result may have been somewhat surprising since only a year previously Poole[38]* demonstrated the simultaneous existence of solutions of period 4π for the Mathieu equation. This can be extended to show coexistence of solutions with period mπ, $m>2$; in fact, it is easily shown[31] that if Mathieu's equation has a periodic solution with a period mπ, $m>2$, then all of its solutions have this period.

Undoubtedly the impossibility of coexistence for the Mathieu equation led Ince astray, for in 1926 Ince[22], [34] followed somewhat later by Markovic[34], published the erroneous statement that equation (1.1), with $\phi(x)$ an even function, cannot have more than one solution of period π or 2π. This error was noted by Borg (see [2], p.90) in his remarkable dissertation where he gives the Meissner equation as a counter-example to disprove the statement of Ince and Markovic.

In the same thesis Borg proved the following deep theorems for equation (1.1):

---

[*] See also [26] where the same result is given for the more general Hill's equation.

Theorem 2.1

A necessary and sufficient condition that all odd intervals of instability shall vanish is that

$$\phi(x + \pi/2) = \phi(x)$$

Theorem 2.2

All intervals of instability, except the zero-th, vanish only if $\phi(x)$ is a constant.

It is somewhat ironical that Ince* furnished a counter-example to his 1926 statement in his second paper on Periodic Lamé Functions. Although he did not state it explicitly, his results show that, for suitable values of the parameters in the Lamé equation, it is possible for all but a finite number of intervals of instability to vanish. Since the Lamé equation has doubly-periodic coefficients, solutions with imaginary periods can occur and can, as was shown by Erdélyi[8], coexist.

The stability question for the Whittaker equation was subjected to an extensive numerical attack by Klotter and Kotowski**. They give stability curves for a range of values. However, for the range of values they examine, only one intersection of an odd and even solution of the same fundamental period (indicating coexistence) is exhibited.

Another equation, which McLachlan[36] calls the Frequency Modulation equation, has been treated by Cambi[4]***. For this equation all even intervals of instability, except the zero-th, can vanish. However, none of the odd intervals will vanish. These results are valid as long as the equation remains non-singular. It may be noted that if the absolute value of the modulation parameter is equal to or greater than unity, the equation becomes singular. In this sense, the Frequency Modulation equation belongs to a different category than the Mathieu or Whittaker equations.

---

* See [24], in particular, paragraph 5, pp. 87-89.

** See [28]. The fundamental periods in this paper are 2π and 4π.

*** Klotter and Kotowski, [28], refer to an unpublished report of A. Weigand on the stability question for this equation.

3. The classification of equations

Equation (1.1) is called Hill's equation by many writers. We shall use the name, Hill's equation, in a more restricted sense to designate the equation

$$(1) \qquad y'' + \left(\sum_{0}^{\infty} A_m \cos^{2m} x\right)y = 0 \quad .$$

This avoids the cumbersome terminology of calling (1) the Even Hill's equation or the Restricted Hill's equation. If, in place of the infinite series, the coefficient of y is a finite polynomial with m terms in $\cos 2x$, then the equation will be called the Finite Hill equation. When $m=1$ it becomes the Mathieu equation, and for $m=2$ it becomes the Whittaker equation. Another equation with periodic coefficients (doubly-periodic, in fact) is the Generalized Lamé equation

$$(2) \qquad y'' + \left(\sum_{0}^{m} A_n \,\mathrm{sn}^{2n} x\right)y = 0 \;,$$

where sn is the elliptic sine function. For $m=1$, equation (2) reduces to the Lamé equation, while for $m=2$, it is the Lamé wave equation.

An equation of considerable importance in this study is

$$(3) \qquad (1 + a \cos 2x)y'' + (b \sin 2x)y' + (c + d \cos 2x)y = 0$$

which does not appear to have been named previously. We propose to name it the Ince equation in honor of the British mathematician who contributed so much to this field. The Ince equation is the most general trigonometric equation leading directly without transformation to three-term recurrence relationships.

It is, of course, possible to continue and list all the equations to be treated in this thesis. A more useful approach is to display the equations in a sequence of three-dimensional manifolds in a manner which indicates their relation to one another and at the same time their position in the general scheme of differential equations. Such a presentation of the various equations may be found in the Appendix.

4. Remarks on the method

The method invented by Ince[18] to show the impossibility of coexistence for the Mathieu equation is simple and direct. It consists essentially of assuming the simultaneous existence of an even and an odd solution with a fundamental period. If these solutions are expressed as infinite series of cosines and sines, respectively, the convergence requirement leads to the condition that the 2 by 2 determinant formed from the nth and (n - 1)th coefficient of each series, as n approaches ∞, must vanish. Each successive determinant is connected by the recurrence relationships obtained by substituting the assumed solutions into the equation. From these relationships it can be shown in the case of the Mathieu equation that the condition for the vanishing of the determinant is that either the sine or the cosine series vanishs identically. Hence, coexistence cannot occur. The same method for the

Lamé equation* reveals the conditions necessary for coexistence. The success of the method depends in each case on the occurrence of three-term recurrence relationships which lead to the simple but necessary connection between successive determinants.

It seemed natural and desirable to exploit this scheme. The procedure adopted is to transform an equation to some form of the Ince equation and then apply the Ince method. Transformation of the dependent or independent variable, or both, may be employed. Transformations of the dependent variable must, of course, be periodic.

A possible extension of the Ince method to the case in which four-term recurrence relationships occur is introduced. The idea is to use 3 by 3 determinants in place of 2 by 2 determinants. The full scope and validity of this extension has not yet been determined.

5. Scope of method

In this paper the two questions of Section 1 are answered affirmatively. The second question is answered by the discussion of the Whittaker equation which is next after the Mathieu equation in the sequence of Finite Hill equations. The Whittaker equation would normally lead to five-term recurrence relationships [28], and hence it is transformed to an equation yielding three-term recurrence relationships. The necessary and sufficient conditions for coexistence of solutions of the Whittaker equation are developed from this transformed equation which is a reduced form of the Ince equation.

The Ince equation, which, as mentioned before, is the most general trigonometric equation leading directly to three-term recurrence relationships, provides the answer to the first question. All of the equations treated previously, including the Whittaker equation, contain two or three arbitrary parameters. The complete Ince equation has four arbitrary parameters and thus represents an important extension. There is also the possibility of treating certain equations with more than four parameters if they can be transformed to the Ince equation and the excess parameters imbedded in the transformations. Equations with Jacobian elliptic functions as coefficients such as the Lamé and the Hermite Elliptic equation* are handled by transforming them into Ince equations.

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* See [25]. The Lamé equation must first be transformed into a special case of the Ince equation.

Some results on the coexistence question are obtained for the general Finite Hill equation. These give an indication of the nature of the solutions connected with coexistence. The Finite Hill equation finds an important application in the study of non-linear oscillations, since the question of stability of periodic solutions for non-linear differential equations reduces to a question of the stability of a Hill's equation (See [42], p.192).

Certain singular equations are discussed briefly and are interesting because they are the equations which arise from the separation of the wave equation in orthogonal coordinate systems. An inherent difficulty is the problem of the existence of even a single periodic solution, since, as is well known, a differential equation with periodic coefficients does not of necessity have periodic solutions. The point of view taken here is to examine the possibility of coexistence, assuming the existence of periodic solutions. Singular equations can lead to two-term and four-term recurrence relationships as well as three-term. Non-singular equations must have recurrence relationships with an odd number of terms.

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* See [12]. The name Hermite Elliptic equation was chosen to avoid confusion with the better known differential equation for the classical Hermite polynomials.

II. The Whittaker Equation

6. Introductory remarks and origin of the equation

This equation was introduced by E.T. Whittaker in his 1915 paper [45], [17] on differential equations whose solutions satisfy homogeneous integral equations. Actually the equation appeared earlier, in 1902, in a memoir by A. Liapounoff [29], as an illustrative example for the general theory developed there. As the standard form of the Whittaker equation, we shall adopt

$$(1) \qquad y'' + (h + 4mq \cos 2x + 2q^2 \cos 4x)y = 0$$

which (with q replaced by $-p/4$) is the equation given by Whittaker. There is some loss of generality in this form for equation (1), since the coefficient of $\cos 4x$ is constrained to be positive. The question of a negative coefficient will be examined in Section 10.

If q goes to zero as m approaches infinity, and at the same time mq remains finite, then (1) becomes the Mathieu equation. Whittaker remarked that the functions which satisfy equation (1) are related to the Mathieu functions in the same manner that the Associated Legendre functions are related to the Bessel functions. Ince pointed out that the Whittaker equation has the same relation to the confluent hypergeometric equation as the Mathieu equation has to the Bessel equation [19]. Additional properties of the Whittaker equation were discussed by Ince [23] in his paper on the real zeros of its solution.

The Mathieu equation, as is well known, arose from the discussion of the vibrations in an elliptic membrane (see [35], and [45], ch. 19). Therefore it is not surprising that the Whittaker equation occurs in the physical problem of the vibrations of an elliptic membrane with a square law decrease in its density radially from the center[21]. Humbert obtained the Whittaker equation from a separation of the Laplace equation (in four dimensions) by introducing hypercylinders formed by three-dimensional confocal paraboloids. The Mathieu equation arises in an analogous manner in three dimensional space from the confocal paraboloids. The Whittaker equation could also be obtained from a separation of the three-dimensional wave equation in a region bounded by confocal paraboloids.*

A more recent application of the Whittaker equation is found in the paper by Chang on the stability of periodic-beam focusing[6], where the stability chart developed by Klotter and Kotowski** is employed.

7. Periodic solutions

If, under the assumption that periodic solutions exist, a trigonometric series is substituted into the Whittaker equation

$$(1) \qquad y'' + (h + 4mq \cos 2x + 2q^2 \cos 4x)y = 0 \quad ,$$

the recurrence relationships which result would have five terms. In order to obtain a sequence of three-term recurrence relationships it is necessary to consider solutions of the form

$$(2) \qquad y = u \exp(q \cos 2x) \ .$$

The periodic solutions of interest are those with period π or 2π, These can be shown[32] to be series of one of the following four types:

---

[*] See [16]. The remark about the wave equation is credited to E.T. Copson.

[**] See [28], p. 155. Chang makes the change in scale required to change the last period from 2π to π.

(Ce) $$u = \sum_{0}^{\infty} A_{2n} \cos 2nx$$

(Se) $$u = \sum_{1}^{\infty} B_{2n} \sin 2nx$$

(Co) $$u = \sum_{0}^{\infty} A_{2n+1} \cos (2n+1)x$$

(So) $$u = \sum_{0}^{\infty} B_{2n+1} \sin (2n+1)x$$

since the recurrence relations for the coefficients in a general Fourier expansion connect only the coefficients appearing in (Ce) etc.

Without loss of generality it may be assumed that for the even solutions Ce and Co

(Ia) $u(0) = 1$ and $u'(0) = 0$

while for the odd solutions Se and So

(Ib) $u(0) = 0$ and $u'(0) = 1$.

The values of the parameters for which these solutions exist provide the boundary curves in the h,q-plane which separate the intervals of stability and instability. Whenever two solutions of period π or 2π exist simultaneously for the same values of the parameters, the corresponding interval of instability vanishes.

The differential equation obtained by employing the transformation (2) is

(3) $$u'' + (a \sin 2x)u' + (b + c \cos 2x)u = 0$$

with

$$(4) \qquad a = -4q; \quad b = h + 2q^2; \quad c = 4(m-1)q \ .$$

The choice of $+q$ in (2) is arbitrary and no restriction is implied for a which may be positive or negative. The positive sign preceding the term $2q^2 \cos 4x$ is, however, essential.

Consider the periodic solutions with period π. By direct substitution of the series (Ce) in the differential equation, the recurrence relations are obtained:

$$(c-2a)A_2 + 2bA_o = 0$$

$$(5) \qquad (c-4a)A_4 + 2(b-4)A_2 + 2cA_o = 0$$

$$(c-6a)A_6 + 2(b-16)A_4 + (c+2a)A_2 = 0$$

and the general term for $n \geq 2$, is :

$$(6) \qquad \left[c-(2n+2)a\right]A_{2n+2} + 2\left[b-(2n)^2\right]A_{2n} + \left[c+(2n-2)a\right]A_{2n-2} = 0 \ .$$

If a periodic solution $Ce(x)$ exists, its Fourier series will be convergent for all complex values of x since Equation (1) has no singularities. From this, it can be derived that

$$(7) \qquad \lim_{n \to \infty} \sqrt[n]{(A_n)} = 0$$

and, in particular, that $A_n \to 0$ as $n \to \infty$.

To decide for which values of the parameters a periodic solution exists is a difficult problem. It can be shown (see [32] for a detailed proof) that for fixed values of m and q (or, alternatively, of a and c) there exists a denumerable infinitude of values of h (or, alternatively, of b) for which (1) (or (3))has a periodic solution of type Ce or of any one of the types Co, Se, and So. The corresponding values of h (or b) are the roots of a transcendental equation which is called the characteristic equation. Its roots are all real and bounded from below, and their only point of accumalation is ∞.

The characteristic equation for b may be written in a form involving a continued fraction*

$$(8) \quad b = \frac{(1/2)(2a-c)(c)/(2)^2}{1 - b/2^2} + \frac{(4a-c)(2a+c)/(16)^2}{1 - b/4^2} + \frac{(6a-c)(4a+c)/(48)^2}{1 - b/6^2} + \dots + \frac{(2na-c)([2n-2]a+c)/[2(2n)(2n-2)]^2}{1 - b/(2n)^2} + \dots .$$

For the series (Se), the recurrence relations are

$$(9) \quad \begin{aligned} &(c-4a)B_4 + 2(b-4)B_2 = 0 \\ &(c-6a)B_6 + 2(b-16)B_4 + (c+2a)B_2 = 0 \\ &(c-8a)B_8 + 2(b-36)B_6 + (c+4a)B_4 = 0 \end{aligned}$$

and the general term for $n \geq 2$ is

$$(10) \quad [c-(2n+2)a]B_{2n+2} + 2[b-(2n)^2]B_{2n} + [c+(2n-2)a]B_{2n-2} = 0.$$

The coefficients of the B's in equation (10) arise from those of the A's in equation (6) by omitting the first one. The characteristic equation for b is

$$(11) \quad b = 2^2 + \frac{2^2(4a-c)(2a+c)/(16)^2}{1 - b/4^2} + \frac{(6a-c)(4a+c)/(48)^2}{1 - b/6^2} + \dots + \frac{(2na-c)[(2n-2)a+c]/[2(2n)(2n-2)]^2}{1 - b/(2n)^2} + \dots .$$

---

[*] The continued fraction converges for finite values of a, b, and c. See [37], paragraph 56, Theorem 41.

If, in (8), the first complete denominator is equated to zero, the resulting equation for b is exactly equation (11).

The continued fractions given by (8) and (11) will terminate for

(12a) $\qquad c = 2na \qquad n > 1$

(12b) $\qquad c = -2na \qquad n > 0$

unless the corresponding complete denominator vanishes simultaneously. It should be noted that equation (8) also may terminate for $n = 0$ (i.e., $c = 0$), and $n = 1$ (i.e., $c = 2a$). When the continued fraction terminates then the equations become algebraic and there are at most n or $n+1$ values of b obtained from (8) and $n-1$ or n values of b obtained from (11) accordingly as c is a positive or negative multiple of a. However, there are also an infinite number of values for b obtained by equating the appropriate complete denominator to zero.

The nature of the periodic solutions with period π of the Whittaker equation may be inferred as an almost immediate consequence of the recurrence relationships and characteristic equations for b. Some of these inferences are contained in the following two theorems:

Theorem 7.1

For the solutions with period π corresponding to Ce, and of the form

$$(13) \qquad y = \left[\sum_0^\infty A_{2n} \cos(2n)x\right] \exp(q \cos 2x) ,$$

we have

(i) if $c = 2na$, $n > 0$,

then either the infinite cosine series in (14) begins with the term in $2n$ (i.e., the first n coefficients vanish identically) and the values of b are obtained by equating the complete n-th denominator of the continued fraction (8) to zero,

or the values of b are obtained from the terminated continued fraction, and have at most n distinct values.

(ii) If $c = -2na$, $n \geq 0$, then

either the cosine series in (14) terminates with the (n+1)-th term and b has at most n+1 distinct values obtained from the terminated continued fraction (9),

or the values of b are obtained by equating the complete (n+1)-th denominator of the continued fraction to zero.

Theorem 7.2

For the solutions with period π, corresponding to Se,

$$(14) \qquad y = \left[\sum_{1}^{\infty} B_{2n} \sin(2n)x\right] \exp(q \cos 2x) .$$

(i) If $c = 2na$, $n > 1$, then

either the infinite sine series in (14) begins with the term in 2n (i.e., the first n-1 coefficients vanish identically) and the values of b are obtained by equating the complete (n-1)-th denominator of the continued fraction (11) to zero,

or the values of b are obtained from the terminated continued fraction and have at most (n-1) distinct values.

(ii) If $c = -2na$, $n \geq 1$, then

either the sine series in (14) terminates with the n-th term and b has at most n distinct values obtained from the terminated continued fraction (12),

or the values of b are obtained by equating the n-th complete denominator of the continued fraction to zero.

Now consider the periodic solutions with period 2π. For the even solution Co, the corresponding recurrence relationships are

$$
\begin{aligned}
&(c-3a)A_3 + [2(b-1) + (c-a)]A_1 = 0 \\
(15)\qquad &(c-5a)A_5 + 2(b-9)A_3 + (c+a)A_1 = 0 \\
&(c-7a)A_7 + 2(b-25)A_5 + (c+3a)A_3 = 0
\end{aligned}
$$

and the general term for $n \geq 1$ is

$$(16)\qquad [c-(2n+3)a]A_{2n+3} + 2[b-(2n+1)^2]A_{2n+1} + [c+(2n-1)a]A_{2n-1} = 0 \;.$$

As before, $\sqrt[n]{A_{2n+1}} \to 0$ as $n \to \infty$ for a periodic solution.

The characteristic equation for b is

$$(17)\qquad b = \frac{1}{2}(a-c+2) + \frac{(3a-c)(a+c)/6^2}{1 - b/3^2} + \cdots$$

$$\cdots + \frac{[(2n+1)a-c][(2n-1)a+c]/[2(2n-1)(2n+1)]^2}{1 - b/(2n+1)^2} + \cdots \;.$$

The recurrence relationships for the odd solution are

$$
\begin{aligned}
&(c-3a)B_3 + [2(b-1)-(c-a)]B_1 = 0 \\
(18)\qquad &(c-5a)B_5 + 2(b-9)B_3 + (c+a)B_1 = 0 \\
&(c-7a)B_7 + 2(b-25)B_5 + (c+3a)B_3 = 0
\end{aligned}
$$

and the general term for $n \geq 1$ is

$$(19)\qquad [c-(2n+3)a]B_{2n+3} + 2[b-(2n+1)^2]B_{2n+1} + [c+(2n-1)a]B_{2n-1} = 0 \;.$$

Again $\sqrt[n]{B_{2n+1}} \to 0$ as $n \to \infty$ for a periodic solution.

The continued fraction for b is

$$(20) \quad b = \frac{1}{2}(c-a+2) + \frac{(3a-c)(a+c)/6^2}{1 - b/3^2} + \dots$$

$$\dots + \frac{[(2n+1)a-c][(2n-1)a+c]/[2(2n-1)(2n+1)]^2}{1 - b/(2n+1)^2} + \dots \; .$$

Except for the leading term, equation (20) is identical to equation (17). These continued fractions will terminate for

$$(21a) \quad c = (2n+1)a, \qquad n \geq 1$$

$$(21b) \quad c = -(2n+1)a, \qquad n \geq 0$$

unless, as before, the corresponding complete denominator vanishes simultaneously. Thus b has at most n or $n+1$ values; or b is obtained by equating the appropriate complete denominator to zero. If $c=a$, that is $n = 0$ in (21a), then equations (17) and (20) are also identical.

The solutions Co and So behave in a manner similar to Ce and Se and exhibit the same dependence on whether c/a is positive or negative. However, since both of the series Co and So go from 0 to ∞, the results can be conveniently summarized in a single theorem.

Theorem 7.3

For the solutions with period 2π, corresponding to Co

$$(22) \quad y = \left[\sum_{0}^{\infty} A_{2n+1} \cos(2n+1)x\right] \exp(q \cos 2x)$$

or corresponding to So

$$(23) \quad y = \left[\sum_{0}^{\infty} B_{2n+1} \sin(2n+1)x\right] \exp(q \cos 2x)$$

(i) If $c = (2n+1)a$, $n \geq 0$, then

either the infinite series in (22) or (23) begin with the term B_{2n+1} (i.e., the first n coefficients vanish identically) and the values of b are obtained by equating the complete n-th denominator of the continued fraction (17) or (20) to zero,

or the values of b are obtained from the terminated continued fraction and have at most n distinct values.

(ii) If $c = -(2n+1)a$, $n \geq 0$, then

either the trigonometric series in (22) or (23) terminates with the $(n+1)$-th term and b has at most $n+1$ distinct values obtained from the terminated continued fraction (18) or (21)

or the values of b are obtained by equating the complete $(n+1)$-th denominator of the continued fraction to zero.

8. Necessary and sufficient conditions for coexistence of periodic solutions

The question of the simultaneous existence of fundamental periodic solutions of the Whittaker equation can be investigated by the method developed by Ince for the Mathieu equation. This method stems from the observation that the general term of the recurrence relationships for Ce is the same (with B written for A) as for Se, and that the one for Co is the same as for So. Rewrite the general term of the recurrence relationships for Ce as

$$(1) \quad [(2n+2)a-c]A_{2n+2}/A_{2n}-[(2n-2)a+c]A_{2n-2}/A_{2n} = 2[b-(2n)^2]$$

and for Se as

$$(2) \quad [(2n+2)a-c]B_{2n+2}/B_{2n}-[(2n-2)a+c]B_{2n-2}/B_{2n} = 2[b-(2n)^2] \; .$$

Equate the left-hand sides of equations (1) and (2); multiply by $A_{2n}B_{2n}$; divide by 2a and rearrange terms to obtain

$$(3)\quad [(n+1)-(c/2a)]\left[A_{2n+2}B_{2n}-B_{2n+2}A_{2n}\right] = -\,[(n-1)+(c/2a)]\left[A_nB_{2n-2}-B_nA_{2n-2}\right]$$

or, rewriting in a convenient notation,

$$(4)\quad \begin{vmatrix} A_{2n+2} & A_{2n} \\ B_{2n-2} & B_{2n} \end{vmatrix} = -\,\frac{(n-1)+(c/2a)}{(n+1)-(c/2a)} \begin{vmatrix} A_n & A_{2n-2} \\ B_n & B_{2n-2} \end{vmatrix} .$$

Equation (4) is a recurrence relation in determinants. Continuing the recursion process and letting

$$(5)\quad K_n = (n-1) + (c/2a) \; ; \qquad H_n = (n+1) - (c/2a)$$

then, after recalling that the first two terms of Ce and the first term of Se are not obtained from (1) and (2), we have

$$(6)\quad \begin{vmatrix} A_{2n+2} & A_{2n} \\ B_{2n+2} & B_{2n} \end{vmatrix} = 2(-1)^{n-1} \prod_{r=1}^{n} (K_r/H_r)A_oB_o \; .$$

Now, from the theory of the Gamma function we have

$$\lim_{N\to\infty} N^{-\lambda}\,\lambda \prod_{n=1}^{N} \left(1+\frac{\lambda}{n}\right) = \frac{1}{\Gamma(\lambda)} \; .$$

Therefore, we find that

$$(7)\quad \lim_{n\to\infty} (-1)^n n^{(c/a)-2} \begin{vmatrix} A_{2n+2} & A_{2n} \\ B_{2n+2} & B_{2n} \end{vmatrix} = -\,\frac{1}{2}\,\frac{\Gamma(2-c/2a)}{\Gamma(c/2a)}\,A_oB_2 \; .$$

Now we know that the periodic solutions of our differential equation will be differentiable any number of times and that, therefore, for any positive integer m

$$\lim_{n\to\infty} n^m A_{2n} = \lim_{n\to\infty} n^m B_{2n} = 0 \; .$$

Consequently, coexistence of periodic solutions can be possible only if

$$(8)\quad \frac{\Gamma(2-c/2a)}{\Gamma(c/2a)}\,A_oB_2 = 0 \; .$$

The case when $a=0$ (with $c \neq 0$, i.e., $m \to \infty$) corresponds to the Mathieu equation. From equation (4) it is seen that for $a=0$ the coefficient of the right-hand determinant is 1. By continuing the recursion process on the right and passing to the limit on the left

$$(9) \qquad A_o B_2 \equiv 0$$

and hence either $A_o = 0$ or $B_2 = 0$. Therefore, either all the A's or all the B's in the trigonometric series vanish identically, indicating (in accordance with the well-known result) that simultaneous periodic solutions can never occur for Mathieu's equation.

With $a \neq 0$, equation (8) is satisfied only if $A_o B_2 = 0$ except when $(c/2a)$ is zero or a negative integer.*

When $A_o B_2 = 0$ then either the cosine series or the sine series vanishes unless

$$(10) \qquad c = 2(n+1)a \qquad\qquad n = 0,1,2,\dots \; .$$

For $n = N$, simultaneous periodic solutions will occur but the leading coefficients of the trigonometric series will vanish up to and including A_{2N} and B_{2N}. The solutions obtained are

$$(11) \qquad u = \sum_{N+1}^{\infty} A_{2n} \cos 2nx$$

$$(12) \qquad u = \sum_{N+1}^{\infty} A_{2n} \sin 2nx \; .$$

For the other alternative, that is

$$(13) \qquad c = -2na \qquad\qquad n = 0,1,2,\dots$$

simultaneous periodic solutions will also occur. For $n=N$, only the leading coefficients up to A_{2N} and B_{2N} will differ. The solutions obtained are

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* Where the gamma-function has simple poles. See [46], p. 236.

$$(14) \qquad u = \sum_{o}^{N} A_n \cos 2nx + \sum_{n+1}^{\infty} A_n \cos 2nx$$

and

$$(15) \qquad u = \sum_{1}^{N} B_n \sin 2nx + \sum_{n+1}^{\infty} A_n \sin 2nx \quad .$$

The values of b for the solutions with period π are obtained by equating the complete $(N+1)$-th denominator of (7.9) to zero, hence

$$(16) \qquad b = (2N+2)^2 + \dots \quad .$$

Similar results are obtained for the solutions Co and So. Rewriting the recurrence relationships ,

$$(17) \qquad [(2n+3)a-c]A_{2n+3}/A_{2n+1} - [(2n-1)a+c]A_{2n-1}/A_{2n+1} = 2[b - (2n+1)^2]$$

$$(18) \qquad [(2n+3)a-c]B_{2n+3}/B_{2n+1} - [(2n-1)a+c]B_{2n-1}/B_{2n+1} = 2[b - (2n+1)^2] \; .$$

As before, equate the left-hand sides, multiply by $A_{2n+1} B_{2n+1}$, divide by $2a$ and rearrange terms to get

$$(19) \qquad [(n+3/2)-(c/2a)]\left[A_{2n+3}B_{2n+1} - B_{2n+3}A_{2n+1}\right] =$$
$$- [(n-1/2)+(c/2a)]\left[A_{2n+1}B_{2n-1} - B_{2n+1}A_{2n-1}\right]$$

or

$$(20) \qquad \begin{vmatrix} A_{2n+3} & A_{2n+1} \\ B_{2n+3} & B_{2n+1} \end{vmatrix} = - \frac{(n-1/2)+(c/2a)}{(n+3/2)-(c/2a)} \begin{vmatrix} A_{2n+1} & A_{2n-1} \\ B_{2n=1} & B_{2n-1} \end{vmatrix} \; .$$

Contimue the recursion process and let

(20) $J_{n+1} = (n-1/2)+(c/2a) \quad ; \quad I_{n+1} = (n+3/2)-(c/2a)$

then

(21) $$\begin{vmatrix} A_{2n+3} & A_{2n+1} \\ B_{2n+3} & B_{2n+1} \end{vmatrix} = 2(-1)^n \prod (J_n/I_n)A_1B_1$$

and as $n \to \infty$

(22) $$[\Gamma(3/2 - c/2a)/\Gamma(-1/2 + c/2a)]A_1B_1 = 0 .$$

Using the same arguments as before, either

(23) $A_1B_1 = 0$ or $(c/2a)-1/2$ is zero or a negative integer.

If $A_1B_1 = 0$, either $A_1 = 0$ and all the A's vanish or $B_1 = 0$ and all the B's vanish. Hence simultaneous periodic solutions do not exist; or

(24) $c = (2n+3)a, \qquad n = 0,1,2,\ldots$

and simultaneous periodic solutions do occur. The leading coefficients vanish as before, and for $n = N-1$ the solutions are

(25) $$u = \sum_{N}^{\infty} A_{2n+1} \cos(2n+1)x$$

and

(26) $$u = \sum_{N}^{\infty} A_{2n+1} \sin(2n+1)x \quad .$$

When the alternative holds, that is

(27) $c = -(2n-1)a , \qquad n = 0,1,2,\ldots$

simultaneous periodic solutions occur with their leading terms differing. These solutions are for $n = N$

(28) $$u = \sum_{o}^{n-1} A_{an+1} \cos(2n+1)x + \sum_{N}^{\infty} A_{2n+1} \cos(2n+1)x$$

$$(29)\quad u = \sum_{0}^{N-1} B_{2n+1} \sin(2n+1)x + \sum_{N}^{\infty} A_{2n+1} \sin(2n+1)x .$$

The values of b for solutions of period 2π are obtained by equating the complete $(N+1)$-th denominator of the continued fraction (7.18) to zero; hence

$$(30)\quad b = (2N+1)^2 + \ldots .$$

As a convenience for future reference, the group u-solution pairs (11)-(12), (14)-(15), (25)-(26), and (28)-(29), will be referred to as Ince-type solutions.

These results may now be summarized in the following:

Theorem 8.1 - The Coexistence Theorem for the Whittaker Equation.

Two periodic solutions of period π or 2π will exist simultaneously if and only if m is a positive or negative integer or zero, and h assumes one of the values h_n which are roots of the characteristic equation for h.

Corollary 8.1

If m is an odd integer, the solutions have period π ; and if m is even, the solutions have period 2π.

Theorem 8.2

If $m = \pm r$, $r = 0,1,2,\ldots$; and $h = h_n$, then the resulting Whittaker equation will have at most either

(i) $r/2$ odd intervals of instability for m even, or

(ii) $(r+1)/2$ even intervals of instability for m odd.

When $m = 0$, the resulting Whittaker equation

$$(31)\quad y'' + (h + 2q^2 \cos 4x)y$$

is a special case of Mathieu's equation. Theorem 8.2 implies that for $m = 0$ all the odd intervals of instability vanish. That this is not a contradiction* may be seen by observing that the least period of this equation (32) is $\pi/2$.

Theorem (8.2) indicates that for $r=1$, all even intervals of instability, except the zero-th, vanish. This corresponds to $m=1$ and $c=0$; and also $m= -1$ and $c= -8q$ (or $c= -2a$). From the recurrence relationship we see that $b=0$ or $h= -2q^2$ is the single characteristic value not contained in the h_r. For all other values of b, two solutions of period π coexist. The corresponding Whittaker equations are

$$(33) \quad y'' + \left[-2q^2 + 4q \cos 2x + 2q^2 \cos 4x\right]y = 0$$

$$(34) \quad y'' + \left[-2q^2 - 4q \cos 2x + 2q^2 \cos 4x\right]y = 0 .$$

This vanishing of all but the zero-th even interval is not a unique occurrence. It happens again, for example, in the case of the Frequency-modulation equation.

9. The impossibility of simultaneous polynomial solutions

The theorems of Section 7 indicate the possibility of solutions with period π or 2π in which the trigonometric series terminate, that is to say, trigonometric polynomial solutions of equation (7.3) may occur.** These polynomial solutions will occur if and only if $c= -na$, where n is any positive integer $0,1,2,\ldots$etc. The coexistence theorem of Section 8 appears to imply the non-existence of simultaneous polynomial solutions,

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[*] See Borg's result, theorem (2.1).

[**] The polynomial solutions are considered in Ince [19].

since the values of b required are not obtained by equating one of the denominators of the continued fraction to zero.

Consider a few particular examples for $c=na$ where equation (7.3) has finite solutions.

If $c=0$, then either $b=0$ or b takes the values given by equation (7.11). If $b=0$, then the equation to be solved is

$$(1) \qquad u'' + (a \sin 2x)u' = 0$$

and the solutions are

$$(2) \quad u = A_0 \qquad \text{and} \qquad u = K \int \exp\left(\frac{a}{2} \cos 2x\right)dx \, .$$

The second solution is, of course, not periodic.

If equation (1) is transformed to the normal form of the Whittaker equation, the degenerate appearance disappears and it becomes

$$(3) \qquad y'' + (-2q^2 + 4q \cos 2x + 2q^2 \cos 4x)y \, .$$

If $c= -2a$, then

$$(4a) \qquad u = A_0 + A_2 \cos 2x \qquad \text{for } b^2 - 4b - 4a^2 = 0$$

and also

$$(4b) \qquad u = B_2 \sin 2x \qquad \text{for } b = 4 \, .$$

If $c = -4a$, then

$$(5a) \qquad u = A_0 + A_2 \cos 2x + A_4 \cos 4x$$

$$\text{for } b = \cfrac{(1/2)(2a-c)(c)/2^2}{1 - b/2^2 + \cfrac{(4a-c)(2a+c)/16^2}{1 - b/4^2}}$$

and also

(5b) $u = B_2 \sin 2x + B_4 \sin 4x$

for $b = 2^2 + \dfrac{2^2(4a-c)(2a+c)/16^2}{1 - b/4^2}$.

If $c = -a$, then

(6a) $u = A_1 \cos x$ for $b = 1 + a$

and also

(6b) $u = B_1 \sin x$ for $b = 1 - a$.

If $c = -3a$, then

(7a) $u = A_1 \cos x + A_3 \cos 3x$

for $b^2 - (10+2a)b - (3a^2-18a-9) = 0$

and also

(7b) $u = B_1 \sin x + B_3 \sin 3x$

for $b^2 - (10-2a)b - (3a^2+18a-9) = 0$.

It is seen that for a given value of c, $(a \neq 0)$, the value of b is different for the even and odd polynomial solutions. It is clear, by examining the continued fractions for the even and odd solutions, that the same pattern will continue for all n.

However, the impossibility of simultaneous polynomial solutions can be demonstrated in a simple and direct manner. Assume v and w are two solutions of equation (7.3) with $v(0)=1$, $v'(0)=0$, $w(0)=0$, and $w'(0)=1$, then

(8) $v'' + (a \sin x)v' + (b + c \cos 2x)v = 0$ and

(9) $w'' + (a \sin x)w' + (b + c \cos 2x)w = 0$.

Multiply (8) by w and (9) by v, and subtract (8) from (9), then

$$(10)\quad vw'' - wv'' + a \sin 2x\,(vw'-wv') = 0 \;.$$

Rewrite as

$$(11)\qquad \frac{vw'' - wv''}{vw' - wv'} = -\,a \sin 2x$$

and integrate. Hence

$$\log(vw'-wv') = \frac{a}{2}\cos 2x - \frac{a}{2}$$

or

$$(12)\begin{vmatrix} v & v' \\ w & w' \end{vmatrix} = \exp(-\,a \sin^2 x) \;.$$

If v and w are both finite trigonometric polynomials, then the left-hand side of equation (12) will also be finite. The right-hand side, for $a \neq 0$, cannot terminate, and, therefore, v and w cannot both be polynomial solutions. Therefore, the transformed Whittaker equation cannot have simultaneous polynomial solutions.

The question of whether the Whittaker equation can have coexisting polynomial solutions will be discussed in Chapter V.

10. Imaginary coefficients and coexistence

If the coefficient of the $\cos 4x$ term in the Whittaker equation is taken to be negative, the equation is

$$(1)\quad y'' + (h + p \cos 2x - 2q^2 \cos 4x) \;.$$

Assuming solutions of the form

$$(2)\quad y = u \exp(i\,q \cos 2x) \;,$$

a differential equation analogous to (7.3) is obtained

(3) $u'' + (a \sin 2x)u' + (b + c \cos 2x)u = 0$

with

(4) $a = -4iq; \quad b = h-2q^2; \quad c = p+4iq$.

From the results obtained in Section 8, it can be concluded that simultaneous periodic solutions of period π or 2π can occur only if

(5) $p = 4miq$

where m is any integer positive or negative.

If q is imaginary, then equation (1) returns to the previous case. However, if q is real, p must be imaginary for coexistence of fundamental periodic solutions.

For Hill's equation

$$y'' + [h + \phi(x)]y = 0 ,$$

where $\phi(x)$ is a real periodic function of x with period π, it is known that periodic solutions can never occur if h is complex or imaginary (see [32], pp. 9-11). This result, however, does not exclude the possibility of periodic solutions if $\phi(x)$ is complex.

The existence of periodic solutions of (1) can be inferred from (2) and (3). Equation (2) shows that the solutions of (1) are periodic whenever u is periodic. From Section 7 it can be seen that the periodicity of the solutions of equation (3) is undisturbed by making $\underline{a}$ and c imaginary, although the coefficients of the infinite series, the A's and B's may, in general, be expected to be complex.

From the tabulation given by Klotter and Kotowski* values of h

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* See [28], Table 3, p.154. Note the factor of 4 required by difference between equation (1) and their equation. The corresponding tabulation for the Whittaker equation (7.1) is given in their Table 2, also on p. 154.

corresponding to the first five periodic solutions of equation (1) can be obtained for a limited range of $2q^2$ (0 to 2) and p (0 to 6). The values of h are real, and range between -2.8 and 6.3 The tables give only positive values for p. However, the stability chart is symmetrical around $p=0$. This fact can be seen at once by making the substitution $x = -\bar{x}$.

11. An extension of the Whittaker equation

The Whittaker equation has three arbitrary parameters. As noted in Section 5 equations with more parameters could be treated if they could be transformed into a Whittaker equation. The extra parameters are absorbed in the transformation. As an example of this, consider the extension of the Whittaker equation

$$(1) \qquad y'' + (4r \sin 2x)y' + (k + p \cos 2x + s \cos 4x)y = 0 .$$

This is readily transformed into the standard form of the Whittaker equation by

$$(2) \qquad y = w \exp(r \cos 2x)$$

and the equation in w is

$$(3) \qquad w'' + \left[(k-2r^2)+(p-4r) \cos 2x +(s+2r^2) \cos 4x\right]w = 0 .$$

Equation (3) is transformed into equation (7.3) by the transformation (7.2) with

$$(4) \qquad q = \pm \left[(s + 2r^2)/2\right]^2$$

while equation (1) is transformed into (7.3) by (7.2) with

$$(5) \qquad q = r \pm \left[(s + 2r^2)/2\right]^2 .$$

An equation similar to (1) which also reduces to the Whittaker equation is

$$(6) \quad y'' + (4r \cos 2x)y' + (h + p \cos 2x - 4r \sin 2x + q \cos 4x)y = 0 .$$

Equations (1) and (6) have four parameters, and combining them leads to an equation with five parameters

$$(7) \quad y'' + (4r \cos 2x + 4s \sin 2x)y'$$
$$+ (h + p \cos 2x - 4r \sin 2x + q \cos 4x + 4sr \sin 4x)y = 0$$

and the transformation to bring equation (7) into a Whittaker equation is

$$(8) \quad y = u \exp(s \cos 2x - r \sin 2x) .$$

Equation (8), with $s = 0$, serves for equation (6).

If $q=2(r^2-s^2)$ in equation (7), then it transforms into the Mathieu equation. If, in addition to $q=2(r^2-s^2)$, $p=4s$, then (7) transforms into the equation of the simple oscillator

$$(9) \quad y'' + ky = 0 .$$

In general, if $P(x)$ is a periodic function, it is always possible to generate a new equation by setting

$$(10) \quad y = u\,P(x)$$

in the Whittaker equation. Obviously equations so generated transform into the Whittaker equation.

The equation

$$(11) \quad y'' + (h + p \sin 2x + q \cos 2x)y = 0$$

is transformed into the Whittaker equation by a simple change of independent variable

(12) $$x = t - \pi/4 .$$

Similarly the equation

(13) $$y'' + (h + p \cos 2x + p \sin 2x + q \sin 4x)y = 0$$

is transformed into the Whittaker equation by

(14) $$x = t - \pi/8 .$$

Although a number of simple transformations have been indicated here, it is not always possible to find a transformation. For example, the equation*

(15) $$y'' + (h + p \cos 2x + q \sin 4x)y = 0$$

cannot be transformed to a Whittaker equation.

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* See [28], p. 150. The least period has been changed from 2π to π.

III. The Ince Equation

12. Introductory remarks

An equation which has four arbitrary parameters and is thus more general than the transformed Whittaker equation (7.3), but which also leads directly without transformation to three-term recurrence relationships, is

$$(1) \quad (1 + a \cos 2x)y'' + (b \sin 2x)y' + (c + d \cos 2x)y = 0 .$$

It will be assumed that $|a| < 1$, setting aside the question of what happens when $|a| \geq 1$, and the equation has singularities on the finite portion of the real axis. With this restriction, (1) is closely connected to some known equations with elliptic functions as their periodic coefficients. This connection will be explored in detail in Chapter IV. As an example, the important Lamé equation can be transformed into a special case of equation (1).*

The Ince equation is interesting because it permits a discussion of the coexistence problem in a case with four arbitrary parameters, which is an extension of previously treated cases. It is not difficult to see by inspection that (1) is the most general non-singular trigonometric equation which yields three-term recurrence formulas.

With certain particular values for the parameters, equation (1) has application to electric circuit theory. For example, with $b=d=0$, the Ince equation becomes the well-known Frequency-modulation equation.

13. Conditions for coexistence of periodic solutions

Again we examine the question of simultaneous existence of periodic solutions by assuming the following four solutions

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* With $b= -a$. See [24], p. 84.

(Ce) $$y = \sum_{0}^{\infty} A_{2n} \cos 2nx$$

(Se) $$y = \sum_{1}^{\infty} B_{2n} \sin 2nx$$

(Co) $$y = \sum_{0}^{\infty} A_{2n+1} \cos(2n+1)x$$

(So) $$y = \sum_{0}^{\infty} B_{2n+1} \sin(2n+1)x \,.$$

Consider the solution Ce. By direct substitution in equation (1), the recurrence relationships are:

$$(\tfrac{d}{2} - b - 2a)A_2 + cA_o = 0$$

(1) $$(\tfrac{d}{2} - 2b - 8a)A_4 + (c - 4)A_2 + dA_o = 0$$

$$(\tfrac{d}{2} - 3b - 18a)A_6 + (c - 16)A_4 + (\tfrac{d}{2} + b - 2a)A_2 = 0$$

and the general term, for $n \geq 2$, is

(2) $$\left[d -(2n+2)b - (2n+2)^2 a\right]A_{2n} + 2\left[c - (2n)^2\right]A_{2n} + \left[d + (2n-2)b - (2n-2)^2 a\right]A_{2n-2} = 0 \,.$$

Now assume a solution Se also exists. Then the corresponding recurrence relationships are:

$$(\tfrac{d}{2} - 2b - 8a)B_4 + (c - 4)B_2 = 0$$

(3) $$(\tfrac{d}{2} - 3b - 18a)B_6 + (c - 16)B_4 + (\tfrac{d}{2} + b - 2a)B_2 = 0$$

$$(\tfrac{d}{2} - 4b - 32a)B_8 + (c - 36)B_6 + (\tfrac{d}{2} + 2b - 8a)B_4 = 0$$

and the general term, for $n \geq 2$, is

$$(4) \quad [d - (2n+2)b - (2n+2)^2 a]^2 B_{2n+2} + 2[c - (2n)^2] B_{2n}$$

$$+ [d + (2n-2)b - (2n-2)^2 a] B_{2n-2} = 0 .$$

Rewrite equations (2) and (4) as

$$[(2n+2)^2 a + (2n+2)b - d] \frac{A_{2n+2}}{A_{2n}} + [(2n-2)^2 a - (2n-2)b - d] \frac{A_{2n-2}}{A_{2n}}$$

$$= 2[c - (2n)^2]$$

$$[(2n+2)^2 a + (2n+2)b - d] \frac{B_{2n+2}}{A_{2n}} + [(2n-2)^2 a - (2n-2)b - d] \frac{A_{2n-2}}{A_{2n}}$$

$$= 2[c - (2n)^2] .$$

Equating the left-hand sides of these equations, clearing fractions and rearranging terms, we get

$$(5) \quad \begin{vmatrix} A_{2n+2} & A_{2n} \\ B_{2n+2} & B_{2n} \end{vmatrix} = \frac{(2n-2)^2 a - (2n-2)b - d}{(2n+2)^2 a + (2n+2)b - d} \begin{vmatrix} A_{2n} & A_{2n-2} \\ B_{2n} & B_{2n-2} \end{vmatrix} .$$

Adopt the notation

$$(6) \quad K_n = (n-1)^2 - (n-1) \frac{b}{2a} - \frac{d}{4a} \qquad \text{and} \qquad H_{n+1} = (n+1)^2 + (n+1) \frac{b}{2a} - \frac{d}{4a}$$

whence

$$(7) \quad \begin{vmatrix} A_{2n+2} & A_{2n} \\ B_{2n+2} & B_{2n} \end{vmatrix} = 2 \prod_{r=1}^{n} (K_r/H_r) A_0 B_2 .$$

Since unlimited differentiability of the trigonometric series requires that, for any $K > 0$,

$$(8) \qquad \lim_{n \to \infty} \begin{vmatrix} A_{2n+2} & A_{2n} \\ B_{2n+2} & B_{2n} \end{vmatrix} n^K = 0$$

equation (8) implies that

$$(9) \qquad \frac{\Gamma(1+p+q)\,\Gamma(1+p-q)}{\Gamma(-p+q)\,\Gamma(-p-q)} A_0 B_2 = 0$$

where $p = b/4a$, $q = (b^2 + 4ad)^{1/2}/4a$.

The coexistence question is settled for the solutions of period π by an examination of equation (9). Briefly stated, unless the argument of one of the two gamma-functions in the denominator of (9) is zero or a negative integer, two solutions of period π cannot exist simultaneously unless $A_0 B_2 = 0$. In this case, the recurrence relations (2) and (4) show, that

$$(2n+2)^2 a + (2n+2) b - d = 0$$

for at least one integer $n = 1,2,3, \ldots$, unless either all $A_{2n} = 0$ or all $B_{2n} = 0$. This is equivalent to stating that at least one of the Gamma factors in the numerator of (9) must be infinite. Altogether, we have as a necessary condition for coexistence of two linearly independent solutions of period π that

$$(10) \quad p \pm q = \pm n, \qquad n = 0,1,2,\ldots$$

The characteristic values for c are taken from the continued fraction obtained in the usual way. In this case, however, there are three possibilities. The values of d may be taken from a terminated continued fraction or from equating the appropriate infinite complete denominator to zero, as in the Whittaker case. In addition, the values of d may be taken from the terminated continued fraction which begins with some nth denominator.

Now consider the solution C_0. Again, by direct substitution, the recurrence relationships are

$$(d-3b-9a)A_3 + [(2c-2)+d-b-a]A_1 = 0$$

$$(d-5b-25a)A_5 + 2(c-9)A_3 + (d+b-a)A_1 = 0 \tag{11}$$

$$(d-7b-49a)A_7 + 2(c-25)A_5 + (d+3b-9a)A_3 = 0$$

and the general term for $n \geq 1$, is

$$\begin{aligned}[d-(2n+3)b-(2n+3)^2a]A_{2n+3} &+ 2[c-(2n+1)^2]A_{2n+1} \\ &+ [d+(2n-1)b-(2n-1)^2a]A_{2n-1} = 0 .\end{aligned} \tag{12}$$

Similarly, for the solution So, the recurrence relationships are

$$(d-3b-9a)B_3 + [2(c-1)-d+b+a]B_1 = 0$$

$$(d-5b-25a)B_5 + 2(c-9)B_3 + (d+b-a)B_1 = 0 \tag{13}$$

$$(d-7b-49a)B_7 + 2(c-25)B_5 + (d+3b-9a)B_3 = 0$$

and the general term for $n \geq 1$, is

$$\begin{aligned}[d-(2n+3)b-(2n+3)^2a]B_{2n+3} &+ 2[c-(2n+1)^2]B_{2n+1} \\ &+ [d+(2n-1)b-(2n-1)^2a]B_{2n-1} = 0 .\end{aligned} \tag{14}$$

Rewriting equations (12) and (14) as

$$[(2n+3)^2a+(2n+3)b-d]\ \frac{A_{2n+3}}{A_{2n+1}} + [(2n-1)^2a-(2n-1)b-d]\ \frac{A_{2n-1}}{A_{2n+1}} = 2[c-(2n+1)^2] ,$$

$$[(2n+3)^2a+(2n+3)b-d]\ \frac{B_{2n+3}}{B_{2n+1}} + [(2n-1)^2a-(2n-1)b-d]\ \frac{B_{2n-1}}{B_{2n+1}} = 2[c-(2n+1)^2] .$$

Equating the left-hand sides of these equations, clearing fractions and rearranging terms, we get

$$(15) \qquad \begin{vmatrix} A_{2n+3} & B_{2n+3} \\ A_{2n+1} & B_{2n+1} \end{vmatrix} = \frac{(2n-1)^2a-(2n-1)b-d}{(2n+3)^2a+(2n+3)b-d} \begin{vmatrix} A_{2n+1} & B_{2n+1} \\ A_{2n-1} & B_{2n-1} \end{vmatrix} .$$

Adopt the notation

$$(16) \qquad J_{n+1} = (2n-1)^2-(2n-1)\frac{b}{a}-\frac{d}{a} \quad \text{and} \quad I_{n+1} = (2n+3)^2+(2n+3)\frac{b}{a}-\frac{d}{a}$$

whence

$$(17) \qquad \begin{vmatrix} A_{2n+3} & B_{2n+3} \\ A_{2n+1} & B_{2n+1} \end{vmatrix} = 2\prod_{1}^{n+1} (J_n/I_n)A_1B_1 .$$

Again, we know that for any $k > 0$

$$(18) \qquad \lim_{n\to\infty} \begin{vmatrix} A_{2n+3} & B_{2n+3} \\ A_{2n+1} & B_{2n+1} \end{vmatrix} n^k = 0$$

and, therefore,

$$(19) \qquad \frac{\Gamma(3/2+p+q)\,\Gamma(3/2+p-q)}{\Gamma(-1/2-p+q)\,\Gamma(-1/2-p-q)} A_1B_1 = 0$$

where again $p = b/4a$, $q = (b^2+4ad)^{1/2}/4a$.

The coexistence question is settled for the solutions of period 2π by an examination of equation (19). Again, unless the argument of one of the four gamma-functions in (19) is zero or a negative integer, two solutions of period 2π cannot exist simultaneously; or what is equivalent

$$(20) \qquad p \pm q = \pm n - 1/2 ; \qquad n = 0,1,2,\ldots .$$

Here again there are three ways in which the values of c may be obtained from the continued fraction, as stated in the remark following equation (10). It may be noted that for the values of c obtained from the terminated continued fraction beginning with the initial terms, coexistence does not occur, but can occur in the other two instances.

Theorem 13.1-The Coexistence Theorem for the Ince Equation

Simultaneous periodic solutions will exist for the Ince equation if and only if*:

(i) For solutions with period π,

$$p + q = \pm n_1 ; \qquad n_1, n_2 = 0,1,2, \ldots$$

or

$$p - q = \pm n_2$$

(ii) For solutions with period 2π

$$p + q = \pm n_1 - 1/2 ; \qquad n_1, n_2 = 0,1,2, \ldots$$

or

$$p - q = \pm n_2 - 1/2 \quad .$$

Corrolary 13.1

There are, in general, three types of simultaneous periodic solutions. (The m used below stands for either $(2n)$ or $(2n+1)$ accordingly as the solutions are of period π or 2π).

(i) If in theorem (13.1) $n < 0$, then the solutions have the form (truncated solutions)

$$y = \sum_{N}^{\infty} A_m \cos mx$$

$$y = \sum_{N}^{\infty} A_m \sin mx$$

---

* It is assumed that c has the required characteristic value.

(ii) and if $n \geq 0$, then the solutions have the form (split solutions)

$$y = \sum_{0}^{N-1} A_m \cos mx + \sum_{N}^{\infty} A_m \cos mx$$

$$y = \sum_{0}^{N-1} B_m \sin mx + \sum_{N}^{\infty} A_m \sin mx$$

(iii) and finally the solutions which result when more than one of the gamma-functions in equations (9) and (19) has a simple pole. If, for the same values of the parameters of the equation,

$$p \pm q = -r-1 \qquad \text{and also} \qquad p \pm q = s \qquad \text{or}$$

$$p \pm q = -r-1/2 \qquad \text{and also} \qquad p \pm q = s - 1/2$$

where r and s are positive integers, then simultaneous polynomial solutions will occur which may be

(a) truncated for $r \leq s$

$$y = \sum_{N}^{M} A_m \cos mx \qquad 0 < N, M < \infty$$

$$y = \sum_{N}^{M} A_m \sin mx$$

(b) split for $r > s$

$$y = \sum_{0}^{n-1} A_m \cos mx + \sum_{N}^{M} A_m \cos mx \qquad 0 < N, M, < \infty$$

$$y = \sum_{0}^{n-1} B_m \sin mx + \sum_{N}^{M} A_m \sin mx \quad .$$

Corollary 13.2

The Ince equation will have at most

(i) $(n+1)$ even intervals of instability if

$$p \pm q = \pm n \, ; \qquad n = 0, 1, 2, \dots$$

(ii) $|n|$ odd intervals of instability if

$$p \pm q = \pm n - 1/2 \qquad n + 0,1,2, \ldots$$

(iii) $(2n+1)$ even and odd intervals of instability (i.e., only a finite number of intervals of instability remain) if simultaneously

$$p \pm q = \pm n, \quad p \pm q = \pm n - 1/2, \quad n = 0,1,2, \ldots .$$

Corollary 13.3

(i) For $d = 0$, all even intervals of instability, except the zero-th interval, vanish.

(ii) For $d = a + b$, all odd intervals of instability vanish.

(iii) For $d = 0$, and $a = -b$, all intervals of instability except the zero-th, vanish.

The discussion of (ii) and (iii) as related to Borg's theorems (2.1) and (2.2) is deferred until Chapter IV.

14. Special cases. The Frequency-modulation equation

An interesting special case of the Ince equation is the Frequency-modulation equation which has important applications in electronic communications.* It is the equation for a series inductance-capacitance electric circuit with no resistance in which the capacitance varies sinusoidally, and has been discussed by McLachlan [36] and treated extensively by Cambi [4], [3]. The equation is written

$$(1) \quad (1 + a \cos 2x)y'' + cy = 0$$

and again the restriction that $|a| < 1$ is imposed to avoid singularities. Cambi considers the cases where $|a| \geq 1$, but a discussion of this will be deferred.

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* The literature is voluminous. See, for example, [5] and [47].

Since equation (1) is the Ince equation with $b = 0$, and $d = 0$, the results for the Ince equation are directly applicable. Using these values for b and d we obtain

Theorem 14.1

In the Frequency-modulation equation, all the even intervals of instability, except the zero-th, always vanish, and none of the odd intervals of instability ever vanish.

The theorem follows at once from (13.7),

$$A_{2n+2} B_{2n} - A_{2n} B_{2n+2} = 0$$

and (13.17) $A_1 B_1 = 0$.

Two additional interesting equations can be evolved from equation (1). They are

$$(2) \quad (1 + a \cos 2x)v'' - (2a \sin 2x)v' + cv = 0$$

and

$$(3) \quad (1 + a \cos 2x)w'' - (4a \sin 2x)w' + (c - 4a \cos 2x)w = 0$$

and are characterized by the fact that if

$$y = F(x,a), \quad \text{then } v = F'(x,a), \quad \text{and } w = F''(x,a) \quad .$$

From the recurrence relationships for the Ince equation, it can be seen that theorem (14.1) also holds for equations (2) and (3). The first recurrence relation for the Ce solution shows that the constant term A_0 which is present for (1) is zero for the derived equations as expected.

Equation (2) may be considered to be the equation for a resistanceless series inductance-capacitance circuit in which the inductance is varying periodically.

A more general form, of which equation (2) is a particular case, is

(4) $(1 + a \cos 2x)y'' + (b \sin 2x)y' + cy = 0$.

Here again, all of the even intervals of instability, except the zero-th interval, always vanish. But the odd intervals can also vanish, since for

(5a) $b = (2n-1)a$ or

(5b) $b = -(2n+3)a$

simultaneous solutions with period 2π can occur. Equation (4) is perhaps the simplest in which it can happen that only a finite number of intervals of instability remain.

The periodic solutions of equation (4), aside from the fact that those with period π occur almost always in even and odd pairs, behave in a way similar to the solutions of the Whittaker equation. The solutions with period π can terminate if

(6a) $b = 2(n-1)a$

and finite polynomial solutions can occur. If

(6b) $b = -2(n+1)a$

the coefficients of the trigonometric series up to and including A_{2n} and B_{2n} can vanish identically.

In the same manner that equation (3) was obtained from equation (2), equation (4) leads to

(7) $(1 + a \cos 2x)u'' + \left[(b-2a) \sin 2x\right]u' + (c + 2b \cos 2x)u = 0$

which is a particular case of the more general Ince equation. Again, if in (4)

$y = G(x)$, then $u = G'(x)$

and the properties of its solutions can be expected to be similar to those of equation (4).

If we set $b=2a$, then (7) reduces to

$$(8) \quad (1 + a \cos 2x)u'' + (c + 4a \cos 2x)u = 0$$

which is a particular case of

$$(9) \quad (1 + a \cos 2x)y'' + (c + d \cos 2x)y = 0 \quad .$$

This last equation is interesting because the criterion for coexistence is simply

$$(10) \quad d = n^2 a$$

and also because in this case the coefficients of A_n and B_n vanish in two of the three recurrence relations in which they appear. The usual alternative conditions (e.g., $b = 2(n-1)a$ and $b + -2(n+1)a$, etc.) occur , so to speak, simultaneously.

Even and odd intervals of instability do not vanish at the same time, since for n even, even intervals of instability vanish; and for n odd, odd intervals of instability vanish.

The Frequency-modulation equation is the particular case of equation (9) for $n = 0$. If we set $c = n^2$ whenever $d = n^2 a$, then the equation (9) reduces to the equation for the simple oscillator, and the solutions are always

$$A_n \cos nx \quad \text{and} \quad B_n \sin nx \ .$$

An indication of a physical application for equation (9) is given in the next paragraph where its connection with equation (15.7) is described.

15. Normal forms

Rewrite the Ince equation as

$$(1) \quad y'' + \frac{b \sin 2x}{1 + a \cos 2x} y' + \left[d/a + \frac{c - (d/a)}{1 + a \cos 2x} \right] y = 0 .$$

Make the transformation

$$(2) \quad y = u \exp\left(- 1/2 \int \frac{b \sin 2x}{1 + a \cos 2x} dx\right)$$

and obtain

$$(3) \quad u'' + \left[d/a + \frac{c - d/a - b}{1 + a \cos 2x} - \frac{(b^2 + 4ab)\sin^2 2x}{4(1 + a \cos 2x)^2} \right] u = 0$$

which, with a little manipulation, becomes

$$(4) \quad (1 + a \cos 2x)^2 u'' + (h + p \cos 2x + q \cos 4x)u = 0$$

where in terms of the parameters of the Ince equation

$$h = (d/2) - (b^2/8) - (ab/2) + c - b$$

$$p = d + (c - b)a$$

$$q = (ad/2) + (b^2/8) + (ab/2) .$$

Equation (4) is the so-called normal form of the Ince equation, and the results for the Ince equation can be applied.

Recalling that

$$- 1/2 \int \frac{b \sin 2x}{1 + a \cos 2x} dx = \log(1 + a \cos 2x)^{b/2a} ,$$

equation (2) may be expressed as

$$(5) \quad u = y(1 + a \cos 2x)^{- b/2a} .$$

Of the equations in Section 14, the Frequency-modulation is in normal form, and the normal forms for the others appear different only in the absence of a parameter. For equation (14.4), where $d = 0$, the normal form is

$$(6) \quad (1 + a \cos 2x)^2 u'' + [(r-s) + ra \cos 2x + s \cos 4x]u = 0$$

with $r = c - b$ and $s = b^2/8 + ab/2$

and the normal form for equation (14.9), where $b = 0$, is

$$(1 + a \cos 2x)^2 u'' + [(d/2 + c) + (d+ca)\cos 2x + (ad/2)\cos 4x]u = 0 .$$

Closely related to the Frequency-modulation equation is the equation

$$(7) \quad y'' + 2Ry' + [S/(1 + a \cos 2x)]y = 0$$

which describes a series electric circuit, with inductance, capacitance, and resistance, in which the capacitance is varying periodically. Applying the transformation

$$(8) \quad y = u \exp \int (-R)dx ,$$

equation (7) becomes

$$(9) \quad (1 + a \cos 2x)u'' + [(S-R^2) - aR^2 \cos 2x]u = 0$$

which is equivalent to equation (14.9) with a modification of the coefficients.

16. Generalizations

A large class of equations with periodic coefficients, which can be transformed into the Ince equation, and whose solutions, consequently, behave in a similar manner, may be generated by the simple expedient of setting

$$(1) \quad y = u\, P(x)$$

where $P(x)$ is a periodic function. The resulting generalized equation is

(2) $$(1 + a\cos 2x)Pu'' + [(b\sin 2x)P + 2(1 + a\cos 2x)P']u'$$
$$+ [(c + d\cos 2x)P + (b\sin 2x)P' + (1 + a\cos 2x)P'']u = 0 .$$

It is relatively easy to obtain equations of the form in (2) from the Ince equation. The difficulty lies in the inverse problem, that is, in recognizing when an equation is in the form of (2), and can therefore be transformed into an Ince equation. For example, if in

(3) $$(1 + a\cos 2x)u'' + (s\sin 4x)u' + (h + p\cos 2x + q\cos 4x + r\cos 6x)u = 0$$

the parameters have the values

$$s = -ab$$
$$h = c - b^2/8 - ab/2$$
(4) $$p = d - b + ab^2/16$$
$$q = b^2/8 - ab/2$$
$$r = -ab^2/16 ,$$

then the transformation

(5) $$u = y\exp(b/2\cos 2x)$$

reduces (3) to an Ince equation.

The scheme of (1) may be employed to introduce additional parameters. For a transformation similar to (5) of the form

(6) $$y = u\exp(-m\cos 2x)$$

we get

(7) $$(1 + a\cos 2x)y'' + [(b-2m)\sin 2x - 2am\sin 4x]u'$$
$$+[h + p\cos 2x + q\cos 4x + r\cos 6x]u = 0$$

with

$$h = c - mb/2 - ma + m^2/2$$

$$p = d - 2m + am^2/4$$

$$q = mb/2 - ma - m^2/2$$

$$r = am^2/4 \quad .$$

An extension of the Ince equation which is useful as well as interesting is

$$(8) \qquad (1 + a \cos 2x)y'' + (b \sin 2x)y' + \left(c + d \cos 2x + \frac{r}{1 + a \cos 2x} \right) y = 0 \; .$$

Comparing this with equation (15.1), it is seen that if equation (8) is put into the same form as (15.1), it will differ only by a term $r/(1 + a \cos 2x)^2$. It is therefore natural to consider using the transformation (15.2). The resulting equation is

$$(9) \qquad u'' + \left[d/a + \frac{c-d/a-b}{1 + a \cos 2x} + \frac{4r-(b^2+4ab)\sin^2 2x}{4(1 + a \cos 2x)^2} \right] u = 0 \quad ,$$

which is almost identical with equation (15.3), and can easily be written in the same form as equation (15.4), the normal form of the Ince equation,

$$(10) \qquad (1 + a \cos 2x)^2 u'' + \left[h + p \cos 2x + q \cos 4x \right] u = 0 \; .$$

This same transformation (15.2) will reduce equation (8) to the Frequency-modulation equation if the parameters in (8) are

$$c = H + (b/2a)^2$$

$$(11) \qquad d = b - (b^2/8a)$$

$$r = (b/2a)^2 + (m^2/8) \quad .$$

Transformations of the Ince equation employing elliptic functions are also interesting, and lead to known equations. Some of these transformations will be discussed in Chapter IV.

IV. The Hermite Elliptic Equation

17. Introductory remarks

The theory of differential equations with doubly-periodic coefficients, developed by Hermite and Picard, preceded the Floquet theory of differential equations with singly-periodic coefficients. Perhaps the best known equation with elliptic functions as coefficients is the Lamé equation, for which there is an extensive literature. Nonetheless, the coexistence question for the Lamé equation was not discussed until 1940 when Ince* transformed it into a trigonometric equation. It is interesting to note that Ince attributes the idea for his paper to a question in a letter of Hermite[12]. The idea is essentially to consider sin n(am z) instead of sn nz as the true analogue of sin nx. As Ince points out, the coexistence question could not be treated using the series of powers of elliptic functions.

The Hermite elliptic equation was introduced in November, 1879, in a letter to E. Heine, and a slightly simpler equation of the same type was treated earlier by Picard[12]. The normal form of this equation was called the Associated Lamé equation by Ince[20]. In his letter to Heine, Hermite also introduces two other equations with elliptic functions as coefficients, the three equations differing only in the coefficient of the first derivative term. All three of these equations are special cases of an equation introduced by Darboux, but usually referred to as Heun's equation. (see [7] and [13], pp. 161-196.)

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* See [24], pp. 83-99. Ince refers to a letter of Hermite to Dini, Oeuvres, v 4, p. 144.

Extensions in two directions are indicated. The first extends the Hermite elliptic equation in a manner analogous to (16.8). This equation has not been identified with a previously discussed equation. The second extends the Lamé equation to the Lamé wave equation or Ellipsoidal wave equation, and to the Ultra-ellipsoidal wave equation. The Ince equation remains the basis for these discussions.

The point of departure in this chapter is different from those in the preceding ones. In the case of the Whittaker equation and the extensions of the Ince equation, the basic tool was the transformation of the dependent variable. In this chapter we employ a transformation of independent variable to change from elliptic functions to trigonometric functions.

18. The Hermite elliptic equation

The Hermite elliptic equation which contains four arbitrary parameters is written

$$(1)\qquad y'' + \frac{2(r+1)k^2 \text{ sn } x \text{ cn } x}{\text{dn } x} y' + [h-(m-r)(m+r+1)k^2 \text{ sn}^2 x]y = 0 .$$

The transformation to the Ince equation is accomplished by letting

$$(2)\qquad x = \text{am}(t,k)$$

$$(3)\qquad dt/dx = (1 - k^2 \sin^2 t)^{1/2}$$

where, of course, $k^2 < 1$.

Since $y' = \dot{y}t'$ and $y'' = \ddot{y}(t')^2 + \dot{y}t''$

where the dots indicate differentiation with respect to t, the equation becomes

$$(4)\qquad (1-k^2 \sin^2 t)\ddot{y} + [(2r+1)k^2 \sin t \cos t]\dot{y} + [h -(m-r)(m+r+1)k^2 \sin^2 t]y = 0 .$$

Equation (4) is the Ince equation

$$(1 + a \cos 2t)y + (b \sin 2y)y + (c + d \cos 2t)y = 0$$

with

$$\begin{aligned} a &= k^2/(2-k^2) \\ b &= (2r+1)k^2/(2-k^2) \\ c &= [2h - (m-r)(m+r+1)k^2]/(2-k^2) \\ d &= (m-r)(m+r+1)k^2/(2-k^2) \end{aligned} \tag{5}$$

and we note that $k^2 < 1$ implies $|a| < 1$ which was the restriction imposed in Section 12.

With these values for the parameters, the p and q of Section 13 become

$$p = (2r+1)/4 \quad \text{and} \quad q = (2m+1)/4 \tag{6}$$

Theorem 18.1

The condition for coexistence of periodic solutions for the Hermite elliptic equation is that $(r+m)$ or $(r-m)$ be an integer. If $(r+m)$ is odd, or $(r-m)$ even, the solutions will have period $2K$, while if $(r+m)$ is even, or $(r-m)$ odd, the solutions will have period $4K$.[*] The Hermite elliptic equation will have only a finite number of intervals of instability if $(r+m)$ and $(r-m)$ are simultaneously odd or simultaneously even, and also if either $(r+m)$ or $(r-m)$ is both even and odd.

If m and r are integers, then $(m+r)$ and $(m-r)$ are odd or even together, and in this event the Hermite elliptic equation will have only a finite number of intervals of instability. In a similar way the other results for the Ince equation may be translated in terms of r and m.

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[*] Corresponding to π and 2π in the trigonometric form of the equation.

The explanation of Corallary 13.3 (iii) is now easy. The criteria $d = 0$, $a = -b$ correspond to $m = r$ and $r = -1$, and therefore (1) becomes the equation for the simple oscillator

$$y + hy = 0,$$

thus satisfying the conditions of Borg's theorem that $\phi(x)$ must be a constant for all intervals of instability except the zero-th to vanish.

If $r = m$, equation (1) becomes the Picard elliptic equation

$$y'' + \frac{2(r+1)k^2 \text{ sn } x \text{ cn } x}{\text{dn } x} y' + hy = 0 \tag{7}$$

which is transformed by (2) into the special case of the Ince equation given in equation (14.4) with

$$\begin{aligned} a &= k^2/(2-k^2) \\ b &= (2r+1)k^2/(2-k^2) \\ c &= 2h/(2-k^2) \text{ .} \end{aligned} \tag{8}$$

Therefore $p - q \equiv 0$ and consequently:

Theorem 18.2

For the Picard elliptic equation, all the even intervals of instability, except the zero-th interval, vanish identically. Odd intervals of instability will vanish if, and only if, r is an interger.

The Frequency-modulation equation (14.1) has an elliptic function analog in the special case of the Picard elliptic equation

$$y'' + \frac{k^2 \text{ sn } x \text{ cn } x}{\text{dn } x} y' + hy = 0 \tag{9}$$

where

$$\begin{aligned} a &= b = k^2/(2-k^2) \\ c &= h/(2-k^2) \text{ .} \end{aligned} \tag{10}$$

Also

$$(11) \qquad r = 0, \quad m = 0$$

which again leads to the results of theorem 14.1.

The normal form of the Hermite elliptic equation is obtained by removing the term in y' by the transformation

$$(12) \qquad y = u \exp\left[- \int \frac{(r+1)k^2 \text{ sn } x \text{ cn } x}{\text{dn } x} dx \right]$$

$$\text{or} \qquad y = u(\text{dn } x)^{-(r+1)}$$

which is the analog of equation (15.2). The equation obtained, analogous to (15.4), is

$$(13) \qquad y'' + [H + p \text{ sn}^2 x + q(\text{cn}^2 x/\text{dn}^2 x)]y = 0$$

which Ince has called the Associated Lamé equation.

An extension of the Hermite elliptic equation to an equation with five parameters is

$$(14) \qquad y'' + \frac{2(r+1)k^2 \text{ sn } t \text{ cn } t}{\text{dn } t} y' + \left[h + p \sin^2 x + \frac{q}{\text{dn}^2 t}\right] y = 0$$

which transforms directly to equation (16.8).

Equation (13) is a special case of the equation

$$(15) \qquad y'' = \left[-h + m(m+1)k^2 \text{ sn}^2 x + \frac{A}{\text{sn}^2 x} + \frac{B \text{ dn}^2 x}{\text{cn}^2 x} + \frac{C k^2 \text{ cn}^2 x}{\text{dn}^2 x}\right] y$$

which Darboux transformed into what is now called the Heun equation.

An interesting equation is

$$(16) \qquad y'' - \frac{k^2 \text{ sn } t \text{ cn } t}{\text{dn } t} y' + \left[a + (b - ak^2)2n^2 t + (c - bk^2)\text{sn}^4 t - ck^2 \text{ sn}^6 t\right] y = 0$$

which transforms into the Whittaker equation,

(17) $\ddot{y} + (a + b \sin^2 t + c \sin^4 t)y = 0.$

For $c = 0$, equation (16) is the analog of the Mathieu equation.

19. The Lamé equation

The Hermite elliptic equation reduces to the Lamé equation when $r = -1$

(1) $y'' + [h - m(m+1)k^2 \operatorname{sn}^2 x]y = 0$.

This equation contains only three arbitrary parameters. The corresponding trigonometric equation is

(2) $(1 - k^2 \sin^2 t)\ddot{y} - (k^2 \sin t \cos t)\dot{y} + |h - m(m+1)k^2 \sin^2 x|y = 0$

which is the Ince equation with

$$a = -b = k^2/(2-k^2)$$

(3) $$c = [2h - m(m+1)k^2]/(2-k^2)$$

$$d = m(m+1)k^2/(2-k^2) \text{ .}$$

The values of p and q are

(4) $p = -1/4$, $q = (2m+1)/4$.

From $p+q = n$ or $p-q = n-1/2$, we get $m = 2n$, and from $p+q = n-1/2$ and $p-q = n$, we get $m = 2n-1$. This provides

Theorem 19.1

If and only if m is an integer, including zero, the Lamé equation will have at most $m+1$ intervals of instability.

Corollary 19.1

If and only if m is an interger, then solutions of the Lamé equation with period $2K$ will coexist, and those with period $4K$ will coexist for the same value of m.

Since the Lamé equation is doubly-periodic, it has fundamental imaginary periods in addition to its real periods. The question of coexistence for the imaginary periods $2iK'$ and $4iK'$ can be settled by making an imaginary transformation of the equation (see [8], and [9], esp. pp. 69-72). Specifically, let

$$(5) \qquad X = i(x - K - iK') , \qquad\qquad H = m(m+1) - h$$

and the equation obtained is identical in form with equation (1)

$$(6) \qquad \frac{d^2y}{dX^2} + |H - m(m+1)k'^2 \, sn^2(X,k')|y = 0$$

where $k'^2 = 1 - k^2$ is the complementary modulus.

The characteristic values of h (and also H) for coexistence, are obtained by equating to zero the appropriate infinite complete denominator of the customary continued fraction. When the continued fraction terminates, Lamé polynomials occur, and these never coexist. Erdélyi has shown that for the same value of h, no two distinct solutions, one with $2K$ or $4K$, and the other with $2iK'$ and $4iK'$ can occur.

20. The Lamé wave equation

$$(1) \qquad \frac{d^2w}{dz^2} + (h + p \, sn^2z + q \, sn^4z)w = 0$$

or, as it is sometimes called, the Ellipsoidal wave equation (see [9], 3, pp. 159-163), is like the Hermite elliptic equation in that it has four arbitrary parameters. A physical application of the Lamé wave equation to the problem of damped electromagnetic eigen-oscillations was suggested by Sommerfeld.*

The method now to be developed applies equally to the

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* See [40], pp. 223-224. Reference is made to the work of M.Abraham, [1].

Ultra-ellipsoidal wave equation,

$$(2)\quad \frac{d^2w}{dz^2} + (h + p\,\mathrm{sn}^2 z + q\,\mathrm{sn}^4 z + r\,\mathrm{sn}^6 z)w = 0$$

and therefore we shall use equation (2) in what follows. Let

$$z = \mathrm{am}\ x$$

or

$$(3)\quad \frac{dx}{dz} = (1 - k^2\sin^2 x)^{1/2}\ .$$

Hence

$$(4)\quad w'' - \frac{k^2\sin 2x}{2(1-k^2\sin^2 x)}\,w' + \frac{h + p\sin^2 x + q\sin^4 x + r\sin^6 x}{1 - k^2\sin^2 x}\,w = 0\ .$$

Now, with an arbitrary constant, let

$$(5)\quad w = u\exp\left(m\int\frac{\sin 2x}{(1 - k^2\sin^2 x)}\,dx\right)$$

which transforms (4) into

$$(6)\quad u'' + \frac{(2m-k^2)\sin 2x}{2(1-k^2\sin^2 x)}\,u' + \frac{\sum_0^4 A_n\sin^{2n}x}{(1-k^2\sin^2 x)^2}\,u = 0\ .$$

The method consists of dividing $(1 - k^2\sin^2 x)^2$ into $\sum_0^4 A_n\sin^{2n}x$ after setting $2m = k^2$. The equation obtained is the Whittaker equation

$$(7)\quad u'' + (a + b\sin^2 x + c\sin^4 x)u = 0$$

where

$$a = h + 1/2$$

$$b = (p-1) - (h+1/2)k^2 - (9k^4/64) + 2k^2(h+1/2)$$

$$c = r/k^2$$

and also $b = 3r - qk^2$.

This method is obviously limited since the five parameters of the equation cannot be handled. In fact, once three of them have been selected the other two are determined. However, it does provide a way to locate some specific instances in which the coexistence question can be discussed.

For the Lamé wave equation, $r = 0$, therefore $c = 0$, and hence equation (7) is the Mathieu equation. In this case the method only indicates cases where coexistence can never occur.

V. Finite Hill Equations

21. Impossibility of coexistence of Ince-type solutions

The solutions of the Ince equation in the case of coexistence are such that the coefficients of the sine series and the coefficients of the cosine series are equal except for at most the first N coefficients, which sometimes all vanish identically. However, for the Whittaker equation, as a result of the transformation required to put it into the form of an Ince equation, these Ince-type solutions do not occur. Instead, the solutions of the Whittaker equation are Ince-type multiplied by an exponential in $\cos 2x$. That Ince-type solutions cannot coexist for a Finite Hill equation is the content of the following theorem.

Theorem 21.1

In the differential equation

$$(1) \quad y'' + (C_0 + C_2 \cos 2x + \ldots, + C_{2m} \cos 2mx)y = 0$$

where $m > 0$ is any integer, two solutions with period π of the form

$$(2a) \quad y_1 = \sum_0^N A_{2n} \cos 2nx + \sum_{n+1}^{\infty} A_{2n} \cos 2nx$$

and

$$(2b) \quad y_2 = \sum_1^N B_{2n} \sin 2nx + \sum_{n+1}^{\infty} A_{2n} \sin 2nx$$

or two solutions with period 2π of the form

$$(3a) \quad y_1 = \sum_0^N A_{2n+1} \cos(2n+1)x + \sum_{n+1}^{\infty} A_{2n+1} \cos(2n+1)x$$

and

$$(3b) \quad y_2 = \sum_0^N B_{2n+1} \sin(2n+1)x + \sum_{n+1}^{\infty} A_{2n+1} \sin(2n+1)x$$

cannot exist simultaneously.

Proof: Assume that equations (2) coexist for $N = 0$. Then by direct substitution in the differential equation we obtain the following pair of identities:

$$(4a)\quad \sum_{0}^{\infty} \Big[(2C_o-8n^2)A_{2n}\cos 2nx + C_2A_{2n}\cos(2n+2)x + \dots + C_{2m}A_{2n}\cos(2n+2m)x$$
$$+ C_2A_{2n}\cos(2n-2)x + \dots + C_{2m}A_{2n}\cos(2n-2m)x\Big] = 0$$

$$(4b)\quad \sum_{1}^{\infty} \Big[(2C_o-8n^2)A_{2n}\sin 2nx + C_2A_{2n}\sin(2n+2)x + \dots + C_{2m}A_{2n}\sin(2n+2m)x$$
$$+ C_2A_{2n}\sin(2n-2)x + \dots + C_{2m}A_{2n}\sin(2n-2m)x\Big] = 0 .$$

The recurrence relations obtained from both (4a) and (4b) will be identical for $n > m$. Hence

$$(5)\quad C_{2m}A_{2n+2m} + \dots + C_2A_{2n+2} + (2C_o-8n^2)A_{2n} + C_2A_{2n-2} + \dots + C_{2m}A_{2n-2m} = 0; \quad (n > m) .$$

Now consider the recurrence relations for $n = m$.

$$(6a)\quad C_{2m}A_{4m} + \dots + C_2A_{2m+2} + (2C_o-8m^2)A_{2m} + C_2A_{2m-2} + \dots + C_{2m-2}A_2 + C_{2m}A_o = 0$$

$$(6b)\quad C_{2m}A_{4m} + \dots + C_2A_{2m+2} + (2C_o-8m^2)A_{2m} + C_2A_{2m-2} + \dots + C_{2m-2}A_2 = 0 .$$

It is clear that $A_o = 0$.

Comparing the relationships for $n = m-1$ yields $A_2 = 0$, since $\sin(-2x) = -\sin 2x$ and $\cos(-2x) = \cos 2x$. Thus

$$(7a)\quad C_{2m}A_{4m-2}+\dots+C_2A_{2m}+\big[2C_o-8(m-1)^2\big]A_{2m-2}+C_2A_{2m-4}+\dots+C_{2m-4}A_2+C_{2m-2}A_o+C_{2m}A_2 = 0$$

$$(7b)\quad C_{2m}A_{4m-2}+\dots+C_2A_{2m}+\big[2C_o-8(m-1)^2\big]A_{2m-2}+C_2A_{2m-4}+\dots+C_{2m-4}A_2 - C_{2m}A_2 = 0 .$$

But $A_o = 0$, and therefore, subtracting (7b) from (7a) gives $2C_{2m}A_2 = 0$.

Continuing this process we get, upon reaching the recurrence relationship for $n = 0$,

$$(8) \quad 0 = A_o = A_2 = A_4 = \dots = A_{2m} .$$

Then with (8) and the recurrence relationship for $n = 1$, $A_{2m+2} = 0$ and so on, until the recurrence relationship for $n = m$, $A_{4m} = 0$ is reached. Therefore

$$(9) \quad 0 = A_o = A_2 = \dots = A_{2m} = \dots A_{4m} .$$

Now consider equation (5) for $n = m+1$

$$(10) \quad C_{2m}A_{4m+2} + C_{2m-2}A_{4m} + \dots + C_{2m-2}A_4 + C_{2m}A_2 = 0$$

and applying equation (9)

$$C_{2m}A_{4m+2} = 0 .$$

Continuing with $n = m+2, m+3, \dots$, it is clear that all the A's must vanish. Hence the theorem is true for $N = 0$.

If $N \neq 0$, compare the recurrence relationships for $n = N+m$. Since by hypothesis $A_{2N} \neq B_{2N}$ but all $A_{2n} = B_{2n}$ for $n > N+1$, we obtain

$$(11a) \quad C_{2m}A_{2N+4m} + \dots + C_{2m-2}A_{2N+2} + C_{2m}A_{2N} = 0$$

$$(11b) \quad C_{2m}A_{2N+4m} + \dots + C_{2m-2}A_{2N+2} + C_{2m}B_{2N} = 0 .$$

Subtracting (11b) from (11a)

$$(12) \quad C_{2m}(A_{2N} - B_{2N}) = 0 \qquad \text{or } A_{2N} = B_{2N}$$

contrary to hypothesis, and the theorem holds for solutions with period π.

If solutions (3) coexist, then for $N = 0$ and $n \geq m$, the recurrence relations are the same for (3a) and (3b), or

$$(13) \quad C_{2m}A_{2n+2m+1} + \dots + 2\left[C_o - (2n+1)^2\right]C_{2n+1} + \dots + C_{2m}A_{2n-2m+1} = 0 .$$

Compare recurrence relationships for $n = m-1$ and find, by the same reasoning as before, that

$$C_{2m}A_1 = 0 .$$

By continuing with the relationships for $n = m-2, m-3, \ldots$ we obtain

$$(14) \quad 0 = A_1 = \ldots = A_{2m-1} .$$

Now, from the recurrence relations for $n = 1$, we get $A_{2m+1} = 0$. Examination of the recurrence relations as n increases makes it clear that all the A's must vanish.

For $N \neq 0$, by comparing the recurrence relationships for $n = N + m$, we find the contradiction

$$(15) \quad C_{2m}(A_{2N+1} - B_{2N+1}) = 0 ,$$

which completes the proof of the theorem.

22. Impossibility of trigonometric polynomial solutions

The impossibility of the coexistence of finite trigonometric polynomial solutions, which is not included in Theorem 21.1, could be demonstrated by using the same procedure as in Section 21. However, it is possible to prove the stronger result:

Theorem 22.1

Finite Hill equations cannot have finite trigonometric polynomial solutions.

Proof: Assume a solution of equation (21.1) exists in the form

$$(1) \quad y = \sum_{0}^{N} A_{2n}\cos 2nx \qquad \text{or} \qquad y = \sum_{1}^{N} A_{2n}\sin 2nx$$

where A_{2N} is the last non-zero coefficient in either of the series (1).

For $n > m$ the recurrence relations (5) hold. Consider the recurrence relation for $n = N+m$, since all $A_{2n} = 0$ for $n \geq N+1$

$$(2) \quad C_{2m}A_{2N} = 0 \qquad \text{or} \quad A_{2N} = 0 \, .$$

This contradicts the hypothesis that A_{2N} is the last non-zero coefficient and proves the theorem for solutions with period π.

Assume that a solution of equation (2.1) exists in the form

$$(3) \quad y = \sum_{0}^{N} A_{2N+1}\cos(2N+1)x \qquad \text{or} \quad y = \sum_{0}^{N} A_{2N+1}\sin(2N+1)x \, ,$$

where A_{2N+1} is the last non-zero coefficient of either of the series (3). Consider the recurrence relation (13) for $n = N+m$. Since all $A_{2n+1} = 0$ for $n > N+1$

$$(4) \quad C_{2m}A_{2N+1} = 0 \qquad \text{or} \quad A_{2N+1} = 0 \, ,$$

contradicting the hypothesis that A_{2N+1} is the last non-zero coefficient. This proves the theorem.

23. Coexistence of periodic solutions

In the discussion of the Whittaker equation it was possible to transform the Whittaker equation using

$$y = u \exp(q \cos 2x)$$

into an equation for which the coexistence question could be completely settled. The same basic idea of transformation enables us to prove the following theorem:

Theorem 23.1

If in the Finite Hill equation (21.1) $y'' + (C_0+C_2\cos 2x + \ldots + C_{2m}\cos 2mx)y = 0$, m is an even integer, then it is always possible to choose the constants C_0, ... C_{2m} in such a manner that fundamental periodic solutions coexist.

We first give an example to illustrate the method more clearly.

Consider the equation

$$(1) \quad y'' + (C_o + C_2 \cos 2x + C_4 \cos 4x + C_6 \cos 6x + C_8 \cos 8x)y = 0 \,.$$

Let

$$(2) \quad y = u \exp \int (a \sin 2x + b \sin 4x)dx$$

and obtain the equation

$$(3) \quad u'' + 2(a \cos 2x + b \cos 4x)u' + (c + d \cos 2x + e \cos 4x)u = 0$$

where

$$(4) \quad \begin{aligned} & 2C_S = b^2 \,, \qquad C_6 = - ab \\ & c = C_o + a^2/2 + b^2/2 \\ & d = C_2 + 2a + ab \\ & e = C_4 - a^2/2 \,. \end{aligned}$$

Assume solutions

$$(5a) \quad u = \sum_o^{\infty} A_{2n} \cos 2nx$$

and

$$(5b) \quad u = \sum_1^{\infty} B_{2n} \sin 2nx$$

and get the following recurrence relations

$$(6) \quad \begin{aligned} & (e-4b)A_4+(d-2a)A_2+2cA_6 = 0 \\ & (e-6b)A_6+(d-4a)A_4+2(c-4)A_2+2dA_o = 0 \\ & (e-8b)A_8+(d-6a)A_6+2(c-16)A_4+(d+2a)A_2+2eA_o = 0 \\ & (e-10b)A_{10}+(d-8a)A_8+2(c-36)A_6+(d+4a)A_4+(e+2b)A_2 = 0 \end{aligned}$$

and the general term for $n \geq 3$

$$(7)\quad [e-(2n+4)b]A_{2n+4}+[d-(2n+2)a]A_{2n+2}+2[c-(2n)^2]A_{2n}$$
$$+[d+(2n-2)a]A_{2n-2}+[e+(2n-4)b]A_{2n-4}=0$$

and also

$$(e-6b)B_6+(d-4a)B_4+[2(c-4)-(e-2b)]B_2=0$$
$$(8)\quad (e-8b)B_8+(d-6a)B_6+2(c-16)B_4+(d+2a)B_2=0$$
$$(e-10b)B_{10}+(d-8a)B_8+2(c-36)B_6+(d+4a)B_4+(e+2b)B_2=0$$

and the general term is for $(n \geq 3)$

$$(9)\quad [e-(2n+4)b]A_{2n+4}+[d-(2n+2)a]A_{2n+2}+2[c-(2n)^2]A_{2n}$$
$$+[d+(2n-2)a]A_{2n-2}+[e+(2n-4)b]A_{2n-4}=0\,.$$

Now if

$$(10)\quad e=(2N+4)b \qquad N=0,1,2,\ldots$$

we can set

$$(11)\quad \begin{array}{lll} A_{2n}+B_{2n}\neq 0 & \text{for} & n \geq N+2 \\ A_{2n}=B_{2n}=0 & \text{for} & n < N+2 \end{array}$$

and the following Ince-type solutions with period π coexist:

$$(12)\quad \sum_{n+2}^{\infty} A_{2n}\cos 2nx$$
$$\sum_{n+2}^{\infty} A_{2n}\sin 2nx\,.$$

Other coexisting solutions can be found for particular values of the parameters, but since our purpose is to demonstrate coexistence, which has been done, these additional solutions will not be elaborated.

In exactly the same way, assume the solutions

$$u = \sum_{0}^{\infty} A_{2n+1}\cos(2n+1)x \tag{13a}$$

$$u = \sum_{0}^{\infty} B_{2n+1}\sin(2n+1)x \tag{13b}$$

and obtain the recurrence relations

$$\begin{aligned}
&(e-5b)A_5+(d-3a)A_3+[2(c-1)+(d-a)]A_1 = 0\\
&(e-5b)B_5+(d-3a)B_3+[2(c-1)-(d-a)]B_1 = 0\\
&(e-7b)A_7+(d-5a)A_5+2(c-9)A_3+[(d+a)+(e-3b)]A_1 = 0\\
&(e-7b)B_7+(d-5a)B_5+2(c-9)B_3+[(d+a)-(e-3b)]B_1 = 0
\end{aligned} \tag{14}$$

and the general term $n \geq 3$

$$\begin{aligned}
&[e-(2n+5)b]A_{2n+5} + [d-(2n+3)a]A_{2n+3} + 2[c-(2n+1)^2]A_{2n+1}\\
&\qquad + [d+(2n-1)a]A_{2n-1} + [e+(2n-3)b]A_{2n-3} = 0
\end{aligned} \tag{15}$$

and the equation for B is identical.

Now if

$$e = (2N+5)b\ , \qquad N = 0,1,2,\ldots \tag{16}$$

we can set

$$\begin{aligned}
&A_{2n} = B_{2n} \neq 0 \qquad \text{for} \quad n \geq 2N+5\\
&A_{2n} = B_{2n} = 0 \qquad \text{for} \quad n < 2N+5
\end{aligned} \tag{17}$$

and the following Ince-type solutions with period 2π coexist:

$$\begin{aligned}
&\sum_{n+2}^{\infty} A_{2n}\cos(2n+1)x\\
&\sum_{n+2}^{\infty} A_{2n}\sin(2n+1)x\ .
\end{aligned} \tag{18}$$

This completes the discussion of the example.

The theorem is proved in exactly the same way. Make the transformation for the general Finite Hill equation

$$(19) \qquad y = u \exp \int (D_1 \sin 2x + \dots + D_r \sin 2rx)dx$$

where $2r = m$, and obtain the equation

$$(20) \qquad u'' + 2(D_1 \sin 2x + \dots + D_r \sin 2rx)u' + (D_o + D_{r+1} \cos 2x + \dots + D_{2r} \cos 2rx)u = 0$$

The general recurrence relationships, with the usual restriction $n = m+r+1$, are for solutions with period π,

$$(21) \qquad [D_{2r} - (2n+2r)D_r] \begin{cases} A_{2n+2r} \\ B_{2n+2r} \end{cases} + \dots + 2[D_o - (2n)^2] \begin{cases} A_{2n} \\ B_{2n} \end{cases} + \dots + [D_{2r} + (2n+2r)D_r] \begin{cases} A_{2n-2r} \\ B_{2n-2r} \end{cases} = 0$$

and also for solutions with period 2π

$$(22) \qquad [D_{2r} - (2n+2r+1)D_r] \begin{cases} A_{2n+2r+1} \\ B_{2n+2r+1} \end{cases} + 2[D_o - (2n+1)^2] \begin{cases} A_{2n+1} \\ B_{2n+1} \end{cases} + [D_{2r} + (2n-2r+1)D_r] \begin{cases} A_{2n-2r+1} \\ B_{2n-2r+1} \end{cases} = 0$$

where the bracket notation indicates that A or B can be used in the same recursion formulas. The criteria for coexistence corresponding to (10) and (16) are

$$(23) \qquad D_{2r} = (2n+2r)D_r$$

and

$$(24) \qquad D_{2r} = (2n+2r+1)D_r \, ,$$

and the corresponding pairs of Ince-type solutions are

$$(25) \qquad u = \sum_{n+r}^{\infty} A_{2n} \cos 2nx$$

$$u = \sum_{n+r}^{\infty} A_{2n} \sin 2nx$$

and

$$(26)\qquad u = \sum_{n+r}^{\infty} A_{2n+1}\cos(2n+1)x$$

$$u = \sum_{n+r}^{\infty} A_{2n+1}\sin(2n+1)x \ .$$

The restriction in Theorem 23.1 that m be an even integer arises because for m odd, a transformation corresponding to equation (19) is not available. The use of an exponential transformation for equations in which m is odd cannot remove the $\cos 2mx$ term from the coefficient of y (or u) without adding a $\cos(2n+2)x$ term in its stead. The conjecture that, since m is odd for Mathieu's equation, equations with m odd do not have coexisting fundamental periodic solutions is interesting, but so far as is known has no factual basis.

VI. Singular Equations

24. Introductory remarks

Physical problems give rise in a natural way to singular equations. The Schroedinger equation, which describes an electron in the electric field of a nucleus, for example, leads directly (see [39], p.2) to the singular equation with periodic coefficients

$$(1)\qquad (1-\cos 2x)y'' + (\sin 2x)y' + (c+d\cos 2x)y = 0.$$

Equation (1) is a particular case of the Ince equation (11.1) with $a = b = 1$, and leads to three-term recurrence relationships.

The investigation of the oscillation of the surface of a lake (or landlocked sea) by Chrystal led to his singular equations which Halm[10] transformed into

$$(2)\qquad (\cos 2x)y'' + (b\sin 2x)y' = (d\cos 2x)y = 0 \ .$$

This is the simplest equation with periodic coefficients in the sense that it yields two-term recurrence relationships. The observation that both equation (1) and equation (2) contain only two parameters leads immediately to a consideration of the equation with three parameters, which also has three-term recurrence relationships, namely:

$$(3) \quad (\sin 2x)y'' + (a + b \cos 2x)y' + (c \sin 2x)y = 0$$

which transforms into the hypergeometric equation.

The Ince equation with $a \geq 1$ is a singular equation with three-term recurrence relationships and four parameters. By choosing $a = 1$, another equation with three-term recurrence relationships and three parameters is easily obtained:

$$(4) \quad (1 + \cos 2x)y'' + (b \sin 2x)y' + (c + d \cos 2x)y = 0 \ .$$

A similar equation can be obtained using $a = -1$.

Rellich (see [39],p.4) remarks that it is sometimes possible to change a singular equation into a regular one by a transformation of the dependent or independent variable, or both; but he further states that this can occur in only a few simple cases. The approach adopted here is to employ trigonometric transformations to exhibit the connection between the singular equations with periodic coefficients and well-known equations in algebraic form. Thus it will be shown that (1) transforms into the Associated Legendre Equation, (2),into the Ultra-spherical or Gegenbauer equation, and (3), into the general hypergeometric equation. Since there exist standard references[30],[9] which completely treat the solutions to hypergeometric equations, only a brief discussion of the nature and existence of solutions will be given.

25. Two-term recurrence relationships. Ultraspherical equation.

As stated above, the equation (24.2), given by Halm, leads to two-term recurrence relationships, and consequently may be considered the simplest equation with periodic coefficients. The transformation to the Ultraspherical equation is accomplished by letting

(1) $t = \sin 2x$

and choosing $b = -4m$ and $d = 4n(n+2m)$. Equation (24.2) then becomes the special hypergeometric equation (see [30], p.76)

(2) $(1-t^2)\ddot{y} - (2m+1)t\dot{y} + n(n+2m)y = 0$

whose solutions are the ultraspherical polynomials* $C_n^m(t)$. Therefore, the solutions of the Halm equation are $C_n^m(\sin 2x)$. For certain values of m these become known orthofonal polynomials, in particular, for

$m=0$, Tchebycheff polynomials of the first kind

$m=1/2$, Legendre polynomials

$m=1$, Tchebycheff polynomials of the second kind where the n are, of course, integers.

It is immediately obvious that for $b=0$ (i.e., $m=0$), n need not be an integer, since in this case equation (7) reduces to the trivial periodic equation (setting $d=k^2 > 0$)

(3) $y'' + k^2 y = 0$.

As would be expected, for $b = -4$, (i.e., $m=1$), the transformation

(4) $y = u/\cos 2x$

brings the Halm equation into the form of equation (3) with the requirement for periodic solutions that $(d+4) = k^2 > 0$.

Two-term recurrence relationships occur also in the equation

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* See also [43], pp. 80-85. Note that Szego uses a slightly different notation.

$$(5) \quad (\sin 2x)y'' + (b \cos 2x)y' + (d \sin 2x)y = 0$$

which can be obtained from the Halm equation (24.2) by the phase-shift transformation

$$(6) \quad X = \pi/4 - x .$$

26. Three-term relationships. The hypergeometric and related equations.

The transformation of equation (24.3) into the hypergeometric equation is accomplished by letting

$$(1) \quad t = \cos^2 x$$

and by setting

$$(2) \quad \begin{aligned} a &= -4C + 2(A+B+1) \\ b &= 2(A+B) \\ c &= -4AB \end{aligned}$$

Then equation (24.3) becomes the hypergeometric equation

$$(3) \quad t(1-t)\ddot{y} + [C-(A+B+1)t]\dot{y} - ABy = 0 .$$

The transformation

$$(4) \quad t = \cos 2x$$

carries equation (24.3) into the Jacobi equation (see [43], p.59)

$$(5) \quad (1-t^2)\ddot{y} + [B-A-(A+B+2)t]\dot{y} + n(n+A+B+1)y = 0$$

with

$$(6) \quad \begin{aligned} a &= 2(A-B \\ b &= 2(A+B+1) \\ c &= 4n(n+A+B+1) . \end{aligned}$$

The first of equations (6) shows the well-known fact that the ultraspherical polynomials are Jacobi polynomials with $A=B$.

The Associated Legendre equation

$$(7) \quad (1-t^2)\ddot{y} - 2t\dot{y} + \left[n(n+1) - \frac{m^2}{1-t^2}\right]y = 0$$

is similarly obtained from equation (24.1), using the transformation

$$(8) \quad t = \cos x$$

with $c = n(n+1) - 2m^2$, and $d = - n(n+1)$.

Equation (24.1) can be brought into normal form*

$$u'' + \left[1/2 - d + 1/4 \cot^2 x + \frac{c+d}{2} \csc^2 x\right]u = 0$$

or

$$(9) \quad u'' + \left[n(n+1) + \frac{1}{2} + \frac{1}{4} \cos \quad + \frac{1}{2} + \frac{1}{4} \cot^2 x - m^2 \csc^2 x\right]u = 0 \ .$$

Using the relation between $\cot^2 x$ and $\csc^2 x$, equation (9) may be rewritten as

$$(9') \quad u'' + \left[n(n+1) + \frac{1}{4} - (m^2 - \frac{1}{4})\csc^2 x\right]u = 0 \ .$$

Equation (9') may be considered as a limiting form of Lamé's equation for $k^2 = 1$.

An interesting equation, in appearance resembling the Whittaker equation except for the coefficient of the second derivative, is

$$(10) \quad (1-\cos 4x)w'' + (h + k \cos 2x + q \cos 4x)w = 0$$

which is a cognate of the hypergeometric equation. It is actually the normal form of equation (24.3) which is obtained by using the transformation

$$(11) \quad 2y = w \exp\left(\int \frac{a + b \cos 2x}{\sin 2x}\, dx\right)$$

---

* See [44], p. 63, where the phase-shifted equivalent of equation (9) is given.

followed by appropriate modification of the constants.

Another equation of the same type is

$$(\cos 2x)w'' + (h + k \cos 2x + q \cos 4x)w = 0 \tag{12}$$

which can be transformed by

$$w = u \exp\left(a \int \tan 2x \, dx\right) \tag{13}$$

into

$$(\cos 2x)u'' + (2a \sin 2x)u' + (b + c \cos 2x)u = 0 \, . \tag{14}$$

It is not difficult to illustrate the remark of Rellich to the effect that sometimes singular equations can be transformed into non-singular equations. For example, the equations

$$\begin{aligned} &u'' + (2 \cot x)u' + (c + d \cos 2x)u = 0 \\ (15)\quad &u'' + (b \sin 2x + 2 \cot x)u' + (c + d \cos 2x)u = 0 \\ &(1 + a \cos 2x)u'' + (b \sin 2x + 2 \cot x) + (c + d \cos 2x)u = 0 \end{aligned}$$

under the transformation

$$y = u \sin x \tag{16}$$

become the Mathieu, the transformed Whittaker and the Ince equations, respectively. A generalization of this is the transformation

$$y = u \sin^r x \cos^s x \tag{17}$$

which transforms the singular equation

$$u'' + \left[\frac{2r \cos x}{\sin x} - \frac{2s \sin x}{\cos x}\right]u' + \left[r(r-1)\,\frac{\cos^2 x}{\sin^2 x} + s(s-1)\,\frac{\sin^2 x}{\cos^2 x} - r(s+1) - s\,(r+1) + h + \phi(x)\right]u = 0 \tag{18}$$

into

$$y'' + [h + \phi(x)]y = 0 \, . \tag{19}$$

27. Four-term recurrence relationships. Description of the method.

The method used to discuss the conditions for coexistence for equations with three-term recurrence relationships consisted of finding a recurring 2x2 determinant which related the coefficients of the two assumed coexisting solutions. As an indication of how this idea might be extended, consider the equation

$$(1) \qquad (\sin 3x + \sin x)w'' + (b \cos 3x + c \cos x)w' + (d \sin 3x + e \sin x)w = 0 .$$

With the condition

$$(2) \qquad 3b - c + d + e = 8 ,$$

it is possible to write the recurrence relations for the solutions

$$(3a) \qquad w = \sum_{0} A_{2n} \cos 2nx$$

$$(3b) \qquad w = \sum_{1} B_{2n} \sin 2nx$$

with n sufficiently large, as

$$(4a) \qquad H_n(A_{2n} + A_{2n-2}) + I_n(A_{2n-2} + A_{2n-4}) + J_n(A_{2n-4} + A_{2n-6}) = 0$$

$$(4b) \qquad H_n(B_{2n} + B_{2n-2}) + I_n(B_{2n-2} + B_{2n-4}) + J_n(B_{2n-4} + B_{2n-6}) = 0$$

or

$$(5) \qquad H_n \left| \frac{A_{2n}+A_{2n-2}}{A_{2n-2}+A_{2n-4}} - \frac{B_{2n}+B_{2n-2}}{B_{2n-2}+B_{2n-4}} \right| = J_n \left| \frac{B_{2n-4}+B_{2n-6}}{B_{2n-2}+B_{2n-4}} - \frac{A_{2n-4}+B_{2n-6}}{A_{2n-2}+A_{2n-4}} \right| .$$

Multiplying (5) by $(A_{2n-2}+A_{2n-4})(B_{2n-2}+B_{2n-4})$, and rewriting in determinant form, we get

$$(6) \qquad \begin{vmatrix} A_{2n}+A_{2n-2} & A_{2n-2}+A_{2n-4} \\ B_{2n}+B_{2n-2} & B_{2n-2}+B_{2n-4} \end{vmatrix} = \frac{J_n}{H_n} \begin{vmatrix} A_{2n-2}+A_{2n-4} & A_{2n-4}+A_{2n-6} \\ B_{2n-2}+B_{2n-4} & B_{2n-4}+B_{2n-6} \end{vmatrix}$$

which can be written as 3x3 determinants

$$(7)\qquad \begin{vmatrix} A_{2n} & A_{2n-2} & A_{2n-4} \\ B_{2n} & B_{2n-2} & B_{2n-4} \\ 1 & -1 & 1 \end{vmatrix} = \frac{J_n}{H_n} \begin{vmatrix} A_{2n-2} & A_{2n-4} & A_{2n-6} \\ B_{2n-2} & B_{2n-4} & B_{2n-6} \\ 1 & -1 & 1 \end{vmatrix}$$

where $H_n = 4n^2 - 2bn - d$

and

$$(8)\qquad J_n = -4(n-3)^2 - 2b(n-3) + d .$$

A similar analysis could be developed for solutions with period π.

If it were known that the equation had periodic solutions with period π, equation (7) could be used to examine the conditions for coexistence.

28. Remarks on the periodic solutions.

The statement that a differential equation with singularities has periodic solutions is meaningful only if the periodic solution is continuous everywhere and has a continuous first derivative, or if the periodic solution is an analytic function. Equation (24.3), which is equivalent to the hypergeometric equation, will have two periodic solutions (in general with singularities) if the solutions are single-valued analytic functions. For this it is necessary but not sufficient for the exponents of the hypergeometric equation to be integers or half integers. Even if the exponents are integers, the solutions may be logarithmic, and periodic solutions will not occur (see [9], pp. 71-73).

In his discussion of the Frequency-modulation equation,

$$(1 + a \cos 2x)y'' + cy = 0$$

Cambi [3] points out that for $a \geq 1$, this becomes the Legendre equation. For $a = 1$, he states that all solutions are stable and of period 2π. However, he does not show that there is even one solution regular at both singularities, although it is a curious fact that the algebraic form of the equation does have two linearly independent solutions which are finite at both finite singularities. As Cambi indicates, the solutions have, in general, first derivatives with logarithmic singularities. Thus the solutions are not periodic in the usual sense.

VII. Conclusion

29. Summary of results.

The coexistence problem for Hill's equation, that is, the problem of establishing the conditions under which two independent solutions of period π or period 2π exist simultaneously for the equation

$$y'' + [h + \emptyset(x)]y = 0$$

where $\emptyset(x)$ is of period π, has been investigated, using recurring 2 by 2 determinants. The determinants are formed by equating the recurrence formulas of the coexisting solutions, and the criteria for coexistence are conditions on the parameters of the equation, determined from these recurring determinants. The method is applicable to equations from which three-term recurrence relationships can be obtained either directly or by transformation of the dependent or independent variable.

The coexistence problem is intimately connected with the stability or instability of the solutions of the equation. The solutions of period π and 2π provide the boundary curves between the intervals of stability and the intervals of instability in the stability diagram. The coexistence of any two of these fundamental periodic solutions implies the vanishing of the corresponding interval of instability.

The previously-known results on coexistence for the Mathieu equation, the Lamé equation, and the Frequency-modulation equation are shown to be special cases of the solution of the coexistence problem for the new equation (with $|a| < 1$)

$$(1 + a \cos 2x)y'' + (b \sin 2x)y' + (c + d \cos 2x)y = 0$$

which we have named the Ince equation, in honor of E. Lindsay Ince, the British mathematician. The Ince equation has four arbitrary parameters, and is thus more general than the previously treated equations which had only two or three arbitrary parameters. The Ince equation is the most general non-singular equation with trigonometric coefficients which leads directly without transformation to three-term recurrence relationships. It is shown that certain equations with elliptic functions as their periodic coefficients, such as the Lamé equation, and the Hermite elliptic equation, can be transformed, by a change of independent variable, into an Ince equation.

It is proved that for coexistence to occur for the Ince equation, it is necessary and sufficient that

$$p \pm q = \pm n \quad \text{or} \quad - 1/2 \qquad m,n = 0,1,2, \dots$$

where $p = b/4a$ and $q = (b^2 + 4ad)^{1/2}/4a$

and c has an appropriate characteristic value.

There are, in general, three types of coexisting periodic solutions which may occur: truncated solutions for which the first N coefficients of the odd and even infinite trigonometric series vanish, split solutions in which the first N coefficients are different for each of the trigonometric series, and polynomial solutions which may be either truncated or split.

If $p \pm q = n$, even intervals of instability will vanish, while if $p \pm q = m-1/2$, odd intervals of instability will vanish. Both even and odd intervals of instability can vanish leaving only a finite number of intervals of instability if $p \pm q = 2n$ and $m - 1/2$, simultaneously.

The simplest example of this is the equation

$$(1 + a \cos 2x)y'' + (b \sin 2x)y' + cy = 0$$

for which all even intervals of instability (except the zero-th) vanish, and for which odd intervals can vanish whenever

$$b = \pm (2n+1)a \ .$$

The Frequency-modulation equation

$$(1 + a \cos 2x)y'' + cy = 0$$

is an example for which all even intervals (except the zero-th) vanish, but for which the odd intervals can never vanish.

These examples illustrate the fact that all even intervals of instability (except the zero-th) vanish for $d = 0$. Similarly for $d = a+b$, all odd intervals of instability vanish, while for $d = 0$ and $a = - b$, all intervals of instability (except the zero-th) vanish. This last case corresponds to a Hill equation with $\phi(x)$ a constant.

If the coexisting solutions obtained for the Ince equation are designated Ince-type solutions, it is proven that Ince-type solutions cannot coexist for the Finite Hill equations

$$y'' + \left(\sum_{0}^{M} C_{2m} \cos 2mx\right)y = 0$$

and also that trigonometric polynomial solutions cannot occur.

If m is an even integer, it is shown that coexistence of fundamental periodic solutions can occur. The solutions are Ince-type multiplied by an exponential function with a periodic exponent.

The particular Finite Hill equation with $m = 2$ is called the Whittaker equation, and is the next in increasing complexity in the sequence of Finite Hill equations after the Mathieu equation. The Whittaker equation is most conveniently written

$$y'' + (h + 4mq \cos 2x + 2q^2 \cos 4x)y = 0 \ .$$

The condition for coexistence is that m be a positive or negative integer or zero, with h the required characteristic value. For q imaginary, the condition for coexistence is unchanged. If m is an even integer, odd intervals of instability will vanish, and conversely, if m is an odd integer, even intervals of instability will vanish.

A number of singular equations were examined and identified with known equations, such as the hypergeometric equation. The idea of using 2 by 2 determinants was extended to a method of employing 3 by 3 determinants, which can be used to treat some particular singular equations leading to four-term recurrence relationships.

Appendix

The Classification of Linear Differential Equations

Linear second order differential equations with rational coefficients can be classified in a systematic way according to the kind and number of singular points of the equations (see [26], p. 494). This scheme stems from the Klein-Bocher discovery[26] in 1894 that many important problems in mathematical physics lead to equations which can be derived from a differential equation with five elementary singularities.* Six equations can be derived from the equations with five elementary singularities by a suitable coalescence of two or more singularities. These derived equations are the equations of Lamé, Mathieu, Legendre, Bessel, Weber(or Hermite) and Stokes (a particular case of Bessel's equation). Ten equations can be derived from the equation with six elementary singularities, fourteen equations from the equation with seven singularities; and this may be continued for any number of elementary singularities. There is no loss of generality if three of the singularities are fixed at the points, 0, 1 and ∞, since any three points can be transformed, without change in the exponents relative to these points, by a projective transformation of the independent variable.

The method of classification developed here is based upon the Klein-Bocher discovery and upon the observation that differential equations

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[*] An elementary singularity is a regular singularity with exponents (relative to the singular point) which differ by 1/2. The coalescence of two elementary singularities gives rise to a regular singularity with an arbitrary exponent differénce, while the coalescence of three or more yields an irregular singularity. Irregular singularities are further characterized by their specie; an irregular singularity of the m-th specie is one generated by $m+2$ elementary singularities.

with periodic coefficients can be transformed into algebraic differential equations by a trigonometric trasnformation (e.g., $t=\cos^2x$) or by an elliptic transformation (e.g., $t=\mathrm{sn}^2x$). These algebraic equations can be derived (by coalescence of singularities) from differential equations with elementary singularities. The purpose of this new classification method is to display the differential equations with periodic coefficients in a manner which shows their relation to one another and at the same time indicates their position in the general scheme of linear second order differential equations.

To accomplish this purpose a sequence of three dimensional manifolds is employed. The accompanying charts show four manifolds of interest, the Spherical, Spheroidal, Ellipsoidal, and Hyper-ellipsoidal. These manifolds are best characterized and more easily understood by referring to the general differential equations corresponding to each manifold. These general differential equations are given in both algebraic and trigonometric form in Table I. Since three singularities may be fixed without loss of generality, three of the singularities are located at the points 0, 1, and ∞. The equations in the spherical and spheroidal manifolds have three singularities*, while those in the ellipsoidal and hyper-ellipsoidal manifolds have four singularities.* This fourth singularity may be located at an arbitrary point and is a parameter or irreducible constant of the equations. The three axes used in the manifolds are θ, r and n; where θ is twice the exponent difference at the origin; r is twice the exponent difference at 1; and n is the index in the coefficient of w. The specie of an irregular singularity at infinity is related to n.

---

* These singularities may be elementary, regular and irregular. It should be noted that the equations in a given manifold are not all derivable from the same equation with only elementary singularities.

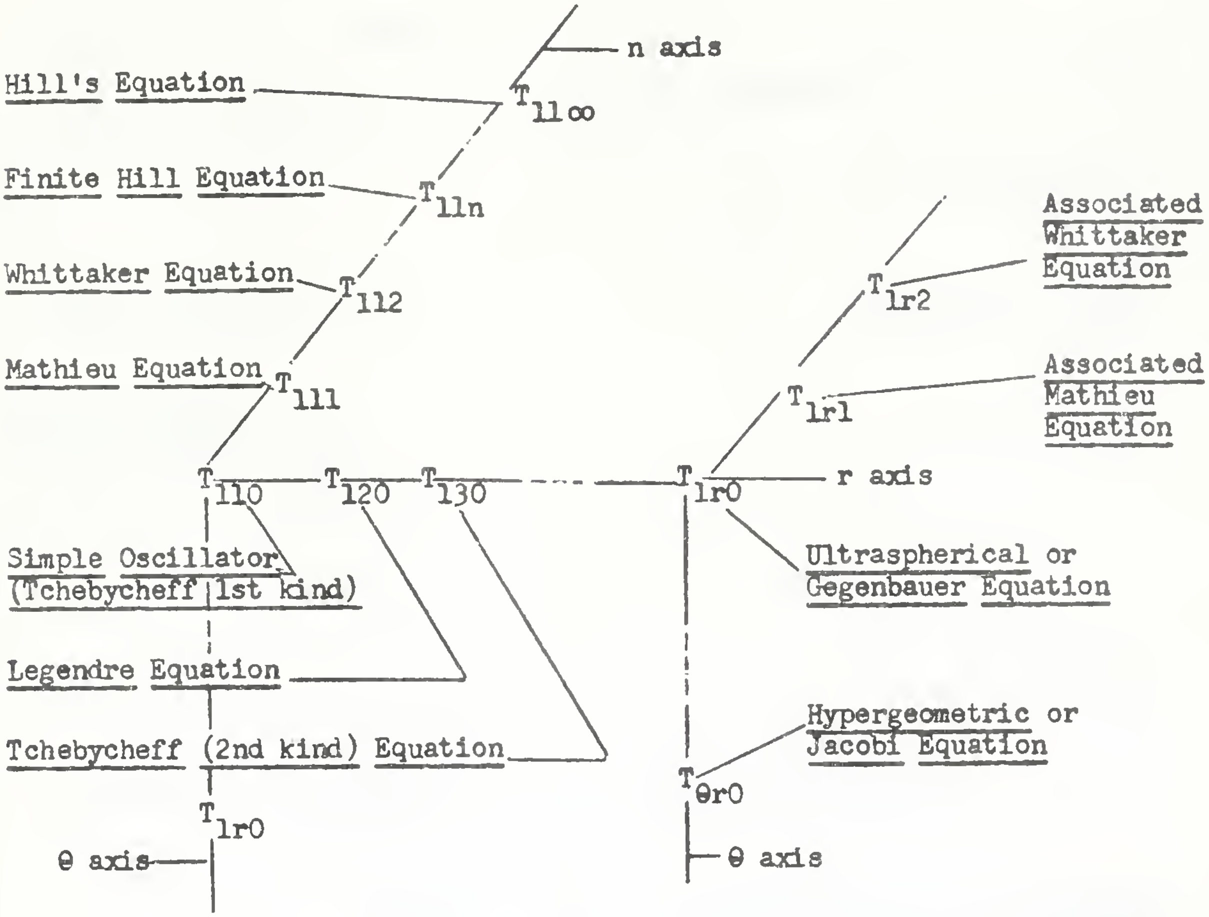
Individual equations are designated by a capital letter and three subscripts, θ, r, and n. T is used for the spherical manifold and S for the spheroidal. E and H are used for the ellipsoidal and hyper-ellipsoidal manifolds, respectively, when the fourth singularity is regular; and L and N, respectively, when the fourth singularity is elementary. In the accompanying charts, many of the equations are named. In general these correspond to the names given in the literature, although a few appear for the first time. Equations $L_{\theta 11}$ and L_{1r1} were given by Hermite in 1880 but are not yet named. In the case of the hyper-ellipsoidal manifold only one equation appears in the literature, N_{112} mentioned by Ince in 1923. The names indicated for N and H equations should be considered tentative. T_{1r2} was named by analogy with T_{1r1}, the Associated Mathieu equation, but as may be noted, is not in consonance with S_{121}, the Associated Legendre equation. Actually S_{121} must be transformed by $t=z^2$ into the Associated Legendre equation, but the connection seems close enough.

From the standpoint of differential equations with periodic coefficients the sequences, T_{11n}, L_{11n}, and E_{11n} appear to be most important. It is interesting to observe that the coexistence problem is solved completely in all three cases for n=1, but for n=2, is solved only in the case of T_{112}. Also, except for some general statements (Borg's theorems) and our results for T_{11n} with n even, little is known for arbitrary n. In terms of the Klein-Bocher classification, equations T_{111} (Mathieu's equation) and L_{111} (Lame's equation) may be derived from the equation with five elementary singularities. Equations T_{112} and E_{111} are derivable from the equation with six elementary singularities. Equation T_{112} has two elementary singularities and one irregular singularity of the n-th specie and therefore may be derived from the equation with $n+4$ elementary singularities.

The major new results of this paper concerned equations T_{112}, E_{111} and T_{11n} (n even). The classification scheme developed here would have served a useful purpose merely by exhibiting the relation of T_{112} and T_{11n} to the hypergeometric equation, and of E_{111} to Heun's equation. By displaying all of the equations discussed in a systematic way, the classification seems to provide a coherence to the discussions.

Chart I

CLASSIFICATION OF EQUATIONS - SPHERICAL MANIFOLD



CLASSIFICATION OF EQUATIONS - SPHERIOIDAL MANIFOLD

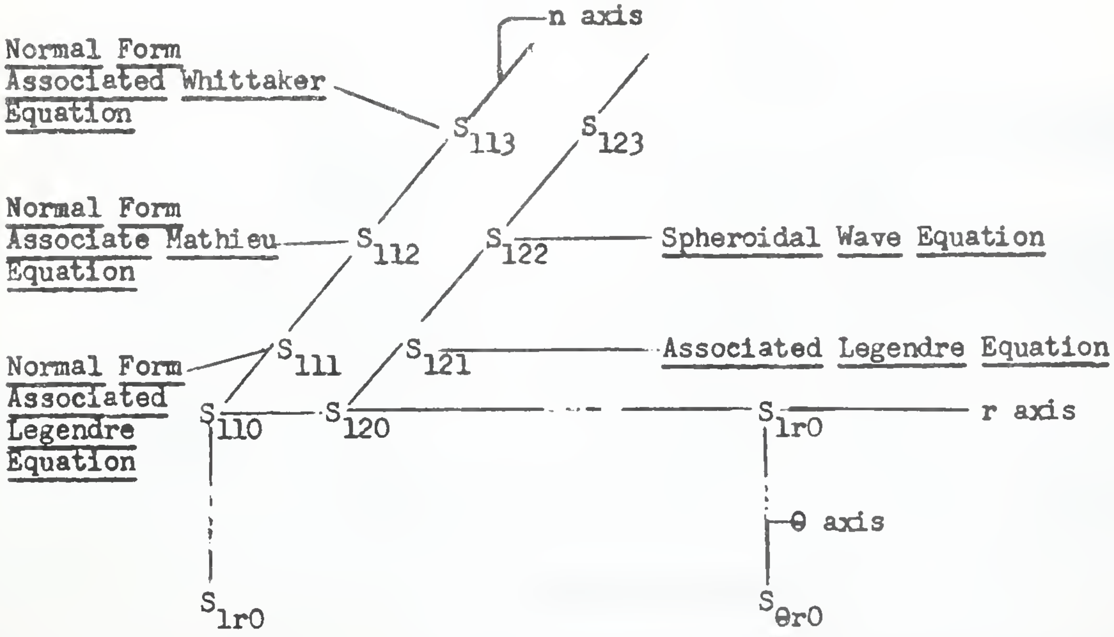
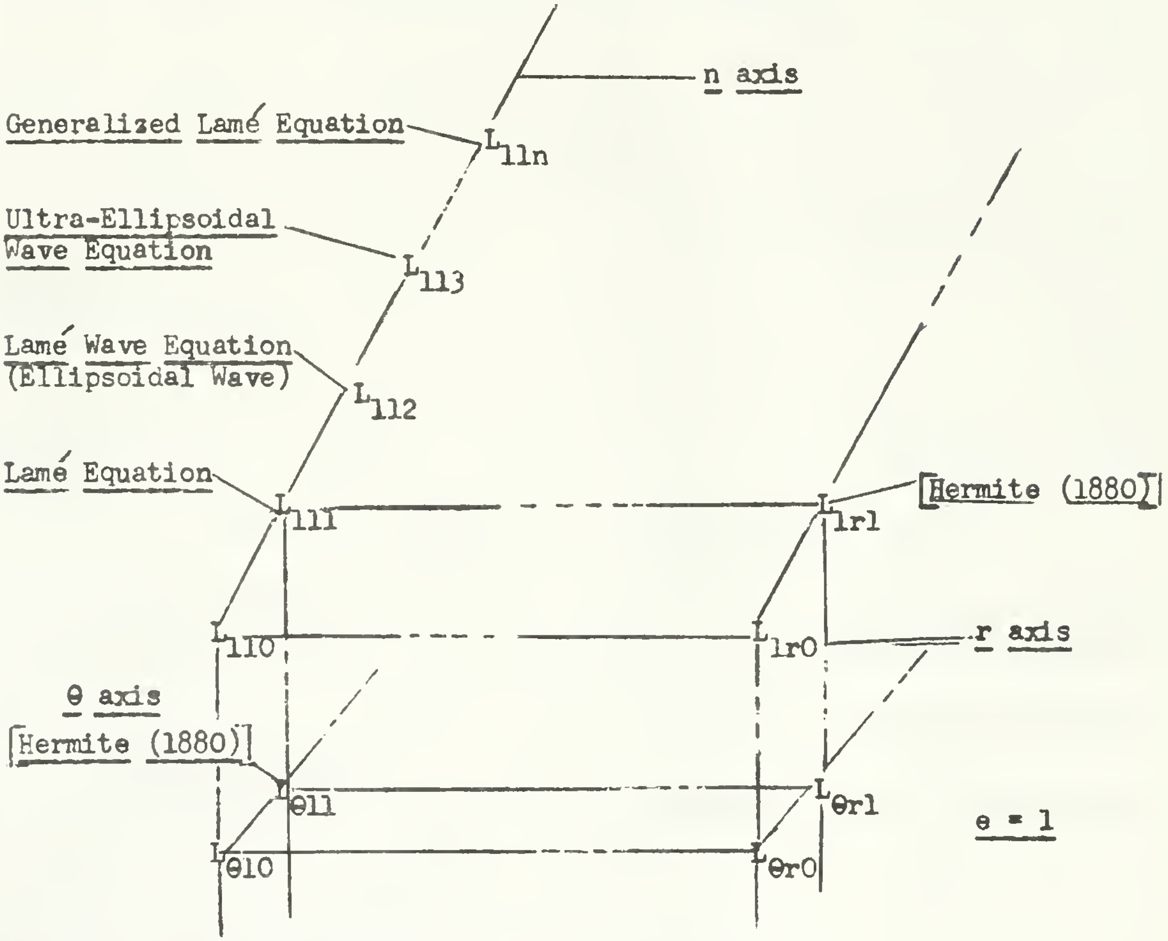
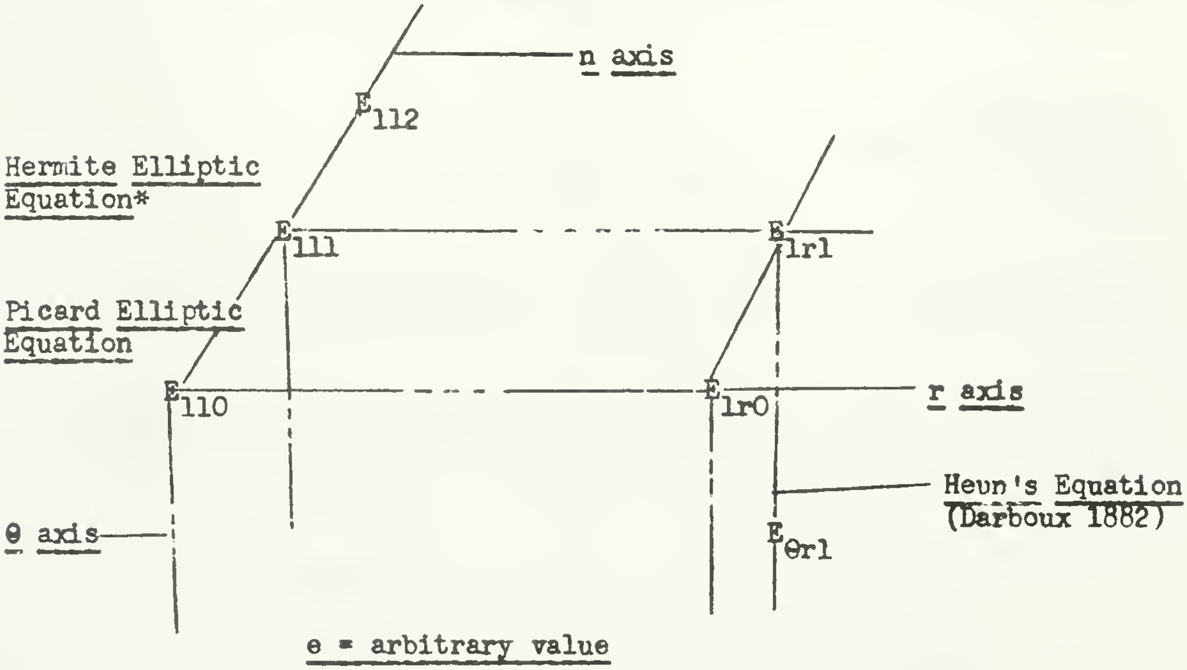

n axis
Normal Form Associated Whittaker Equation
S_{113}
S_{123}
Normal Form Associate Mathieu Equation
S_{112}
S_{122}
Spheroidal Wave Equation
Normal Form Associated Legendre Equation
S_{111}
S_{121}
Associated Legendre Equation
S_{110}
S_{120}
S_{1r0}
r axis
θ axis
S_{1r0}
$S_{\theta r0}$

Chart II

CLASSIFICATION OF EQUATIONS - ELLIPSOIDAL MANIFOLD
(Exponent difference of singularity at a equal to 1/2)



CLASSIFICATION OF EQUATIONS - ELLIPSOIDAL MANIFOLD
(Exponent difference of singularity at a arbitrary, equal to e/2)



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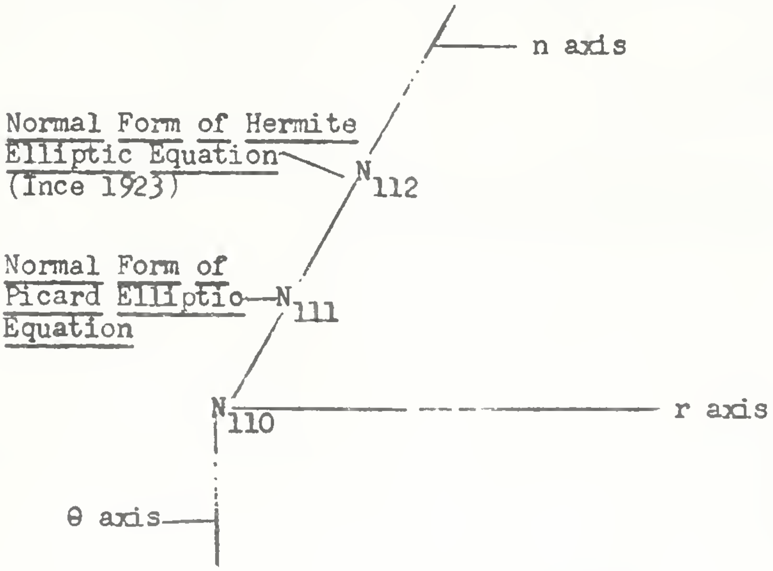
* Transforms to the Ince equation.

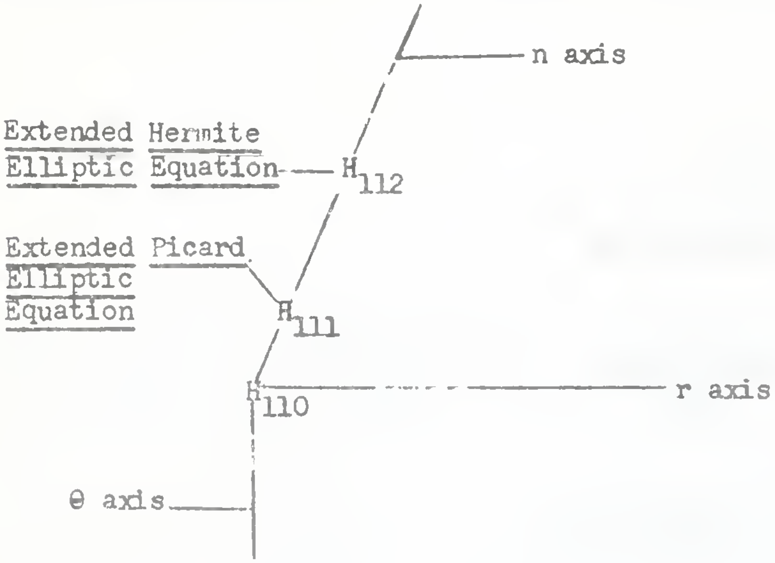
Chart III

CLASSIFICATION OF EQUATIONS - HYPER-ELLIPSOIDAL MANIFOLD
(Exponent difference of singularity at a equal to 1/2)


n axis
Normal Form of Hermite Elliptic Equation (Ince 1923)
N_{112}
Normal Form of Picard Elliptic Equation
N_{111}
N_{110}
r axis
θ axis

e = 1

CLASSIFICATION OF EQUATIONS - HYPER-ELLIPSOIDAL MANIFOLD
(Exponent difference of singularity at a arbitrary, equal to e/2)


n axis
Extended Hermite Elliptic Equation
H_{112}
Extended Picard Elliptic Equation
H_{111}
H_{110}
r axis
θ axis

e = arbitrary value

Table I - Classification of Equations - Algebraic and Transcendental Forms

SPHERICAL - $T_{\theta rn}$:

$$\ddot{w} + \left[\frac{\theta}{t} + \frac{r}{(t-1)}\right]\frac{\dot{w}}{2} + \frac{\sum_{0}^{n} A_n t^n}{4t(t-1)}\, w = 0$$

$$w'' + \left[(r-1)\cot x - (\theta-1)\tan x\right]w' - \left[\sum A_n \cos^{2n} x\right]w = 0$$

SPHEROIDAL - $S_{\theta rn}$:

$$\ddot{w} + \left[\frac{\theta}{t} + \frac{r}{t-1}\right]\frac{\dot{w}}{2} + \frac{\sum_{0}^{n} A_n t^n}{4t(t-1)^2}\, w = 0$$

$$w'' + \left[(r-1)\cot x - (\theta-1)\tan x\right]w' + \frac{\left[\sum A_n \cos^{2n} x\right]}{\sin^2 x}\, w = 0$$

ELLIPSOIDAL - $E_{\theta rn}$:

$$\ddot{w} + \left[\frac{\theta}{t} + \frac{r}{t-1} + \frac{e}{t-a}\right]\frac{\dot{w}}{2} + \frac{\sum_{0}^{n} A_n t^n}{4t(t-1)(t-a)}\, w = 0$$

$$w'' + \left[\frac{(\theta-1)\,\mathrm{en}x\,\mathrm{dn}x}{\mathrm{sn}x} - \frac{(r-1)\,\mathrm{sn}x\,\mathrm{dn}x}{\mathrm{en}x} - \frac{(e-1)k^2\,\mathrm{sn}x\,\mathrm{en}x}{\mathrm{dn}x}\right]w' - k^2\left[\sum A_n \mathrm{sn}^{2n} x\right]w = 0$$

Note: For $e = 1$, $E_{\theta rn}$ becomes $L_{\theta rn}$

HYPER-ELLIPSOIDAL - $H_{\theta rn}$:

$$w'' = \left[\frac{\theta}{t} + \frac{r}{t-1} + \frac{e}{t-a}\right]\frac{w}{2} + \frac{\sum_{0}^{n} A_n t^n}{4t(t-1)(t-a)^2} w = 0$$

$$w'' + \left[\frac{(\theta-1)\,\mathrm{en}x\,\mathrm{dn}x}{\mathrm{sn}x} - \frac{(r-1)\,\mathrm{sn}x\,\mathrm{dn}x}{\mathrm{en}x} - \frac{(e-1)k^2\,\mathrm{sn}x\,\mathrm{en}x}{\mathrm{dn}x}\right]w' - \frac{k^2}{\mathrm{dn}^2x}\left[\sum A_n \mathrm{sn}^{2n}x\right]w = 0 .$$

Note: For $e=1$, $H_{\theta rn}$ becomes $N_{\theta rn}$.

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